# CE 273
# Markov Decision Processes

Lecture 3
## Limiting Behavior of DTMCs

# Previously on Markov Decision Processes

Let $\{X_n, n \geq 0\}$ be a DTMC on $S = \mathbb{Z}^+$ with transition matrix $P$ and initial distribution $a$. For a given $n$, the marginal distribution of $X_n$ is

$$\begin{aligned} a_j^{(n)} &= \mathbb{P}\big[X_n = j\big] \, \forall j \in S \\ &= \sum_{i \in S} \mathbb{P}\big[X_n = j | X_0 = i\big] \mathbb{P}\big[X_0 = i\big] \text{ (Law of Total Probability)} \\ &= \sum_{i \in S} a_i p_{ij}^{(n)} \end{aligned}$$

where $p_{ij}^{(n)}$ is the probability of going from $i$ to $j$ in exactly $n$ steps. Define the $n$-step transition matrix $P^{(n)}$ as

$$P^{(n)} = \left[p_{ij}^{(n)}\right]_{|S| \times |S|}$$

Hence, to compute the marginal distributions, we need to compute the $n$-step transition matrices.

# Previously on Markov Decision Processes

We might sometimes be interested in the expected amount of time spent by the system in different states up to time $n$ (e.g., parking).

Such metrics are called occupancy times. Let $V_j^{(n)}$ be the number of visits to $j$ over $\{0, 1, \ldots, n\}$. Mathematically, occupancy time of $j$ up to time $n$ starting from $i$ is

$$m_{ij}^{(n)} = \mathbb{E}\big[V_j^{(n)} | X_0 = i\big], \forall\, i, j \in S, n \geq 0$$

The matrix of $m_{ij}^{(n)}$ values, also called the occupancy time matrix, is represented by

$$M^{(n)} = \big[m_{ij}^{(n)}\big]_{|S| \times |S|}$$

The occupancy times matrix can be computed from the transition matrix!

# Previously on Markov Decision Processes

Intuitively, to go from $i$ to $j$ in $n$ steps, we need to transition from $i$ to some state $r$ in $k$ steps and from $r$ to $j$ in remaining $(n-k)$ steps.

**Theorem (Chapman-Kolmogorov Equations)**

*The n-step transition probabilities satisfy*

$$p_{ij}^{(n)} = \sum_{r \in S} p_{ir}^{(k)} p_{rj}^{(n-k)}, \forall\, i, j \in S, 0 \leq k \leq n$$

Applying the CK equations recursively, $P^{(n)} = P^n$

**Theorem**

*Let* $P^0 = I$. *For a fixed n,* $M^{(n)} = \sum_{r=0}^{n} P^r$

# Previously on Markov Decision Processes

To study this new classification, let's first define a random variable called the **passage time**

$$\tilde{T}_i = \min\{n > 0 : X_n = i\}$$

It represents the time step when the process visits $i$ for the first time (ignoring the initial state). What is the support of $\tilde{T}_i$?

Given a random variable, we are typically interested in its pmf and expected value. In the context of DTMCs, the following functions are of interest.

1. **Probability that the return time is finite**

$$\tilde{u}_i = \mathbb{P}\big[\tilde{T}_i < \infty | X_0 = i\big]$$

2. **Expected return time**

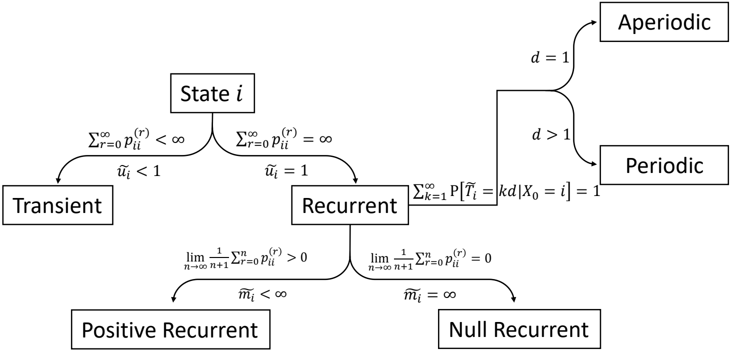
$$\tilde{m}_i = \mathbb{E}\big[\tilde{T}_i | X_0 = i\big]$$

**Interpretation**

$\tilde{u}_i$ can also be viewed as the probability with which $i$ is revisited and $\tilde{m}_i$ is the expected time between consecutive visits.

# Previously on Markov Decision Processes

- Accessibility ($i \to j$)
- Communicating ($i \leftrightarrow j$)
- Communicating Class (All states communicate and the set is maximal)
- Closed Communicating Class ('Blackhole')
- Irreducibility (The entire DTMC is a 'blackhole')



# Lecture Outline

1. Motivating Examples
2. Limiting Behavior of Irreducible DTMCs
3. More Examples
4. Limiting Behavior of Reducible DTMCs

# Motivating Examples

In this lecture, we will extend our transient behavior analysis to cases where $n$ is very large. As mentioned earlier, we are interested in

$$\lim_{n\to\infty} p_{ij}^{(n)}$$

$$\lim_{n\to\infty} \frac{m_{ij}^{(n)}}{n+1} \text{(Cesàro limit)}$$

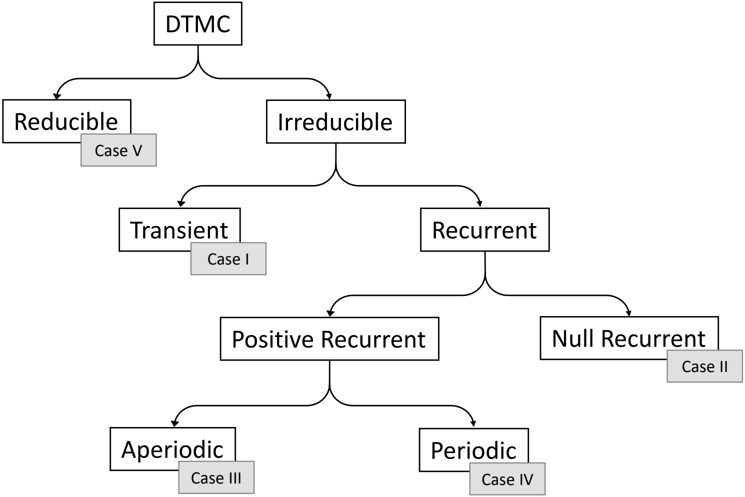
The first limit is called the **limiting probability distribution** (since it helps understand $X_n$ as $n \to \infty$). and the second one is the **limiting occupancy distribution**, i.e., limiting fraction of time spent in $j$ starting from $i$.

It turns out that these limits depend on the type of DTMC. For this reason, we will break down today's lecture into **five cases**.

The classification of states that we saw in last class will help us in this effort. Let's first look at an example for each of the five cases.

# Motivating Examples

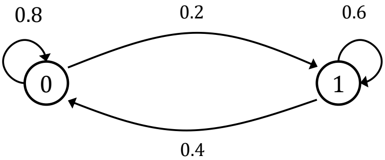
## Classification



# Motivating Examples

## Case III: Aperiodic, Positive Recurrent, Irreducible DTMCs

This is an example of a "well-behaved" case. Consider the following two-state DTMC. Is it irreducible, positive recurrent, and aperiodic?




$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

It is easy to show (or check numerically) that

$$\lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} \frac{M^{(n)}}{n+1} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \end{array}$$

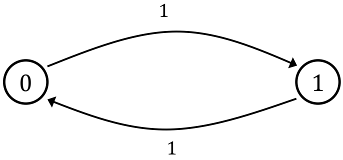
**Observations**:

- Both limits are the same
- Rows in the matrix are identical and add up to 1 (What does it imply?)

# Motivating Examples

## Case IV: Periodic, Positive Recurrent, Irreducible DTMCs

Now consider a periodic two-state DTMC. Is it irreducible and positive recurrent?




$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Find $P^2$ and $P^3$? $\lim_{n\to\infty} P^{(n)}$ does not exist because the $P^{(n)}$ exhibit oscillatory behavior.

$$P^{(2n)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad P^{(2n+1)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

However, limit of $M^{(n)}/(n+1)$ exists.

$$\lim_{n\to\infty} \frac{M^{(n)}}{n+1} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

# Motivating Examples

Case IV: Periodic, Positive Recurrent, Irreducible DTMCs

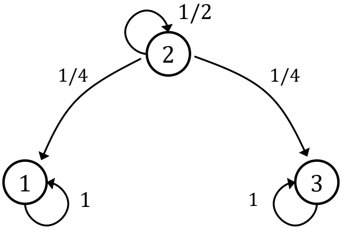
**Observations**:

- $\lim_{n\to\infty} P^{(n)}$ does not exist but $\lim_{n\to\infty} \frac{M^{(n)}}{n+1}$ exists
- All rows of $\lim_{n\to\infty} \frac{M^{(n)}}{n+1}$ matrix are identical and add up to 1

# Motivating Examples

## Case V: Reducible DTMCs

Imagine a 3-state reducible DTMC shown below




$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

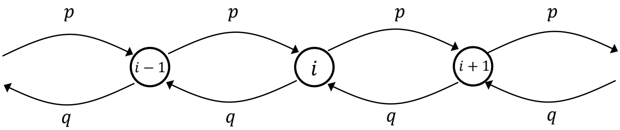
It is easy to show (or check numerically) that

$$\lim_{n\to\infty} P^{(n)} = \lim_{n\to\infty} \frac{M^{(n)}}{n+1} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

**Observations:**

- Both limits are the same
- Rows in the matrix are **not identical** but add up to 1

Consider the following simple random walk. Is this DTMC irreducible?




The above Markov chain can be recurrent or transient depending on the value of $p$. Recall that recurrence, transience, and periodicity are class properties.

Earlier, we showed that

$$p_{00}^{(2n+1)} = 0$$
$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n$$

As $n \to \infty$, $n!$ can be approximated as $\sqrt{2\pi} n^{n+1/2} e^{-n}$ (Stirling's formula).

$$\begin{aligned} p_{00}^{(2n)} &= \frac{(2n)!}{n!n!} p^n q^n \\ &= \frac{(4pq)^n}{\sqrt{\pi n}} \\ &= \frac{(4p(1-p))^n}{\sqrt{\pi n}} \\ &\leq \frac{1}{\sqrt{\pi n}} \end{aligned}$$

Thus, $\lim_{n\to\infty} p_{00}^{(2n)} = 0$ and hence $p_{00}^{(n)} \to 0$. One can also show that $p_{ij}^{(n)} \to 0$ and $\frac{m_{ij}^{(n)}}{n+1} \to 0$.

Note that row sums of $P^{(n)}$ are 1 but not of $lim_{n\to 0} P^{(n)}$.

Is the above DTMC transient or recurrent? Using the alternate criteria,

- It is transient if $\sum_{n=0}^{\infty} p_{00}^{(n)} < \infty$
- It is recurrent if $\sum_{n=0}^{\infty} p_{00}^{(n)} = \infty$

According to the ratio test, $\sum a_n$ converges if $a_{n+1}/a_n \to a (< 1)$. Thus the series $\sum \frac{(4p(1-p))^n}{\sqrt{\pi n}}$ converges when $p \neq 1/2$ and the DTMC is transient.

When $p = 1/2$, since $\sum 1/\sqrt{n}$ diverges, the DTMC is recurrent. In fact, it can be shown to be null recurrent.

For 2D and 3D random walks with $p = 1/4$ and $1/6$, the problem can be reduced to verifying if $\sum (1/\sqrt{n})^2$ and $\sum (1/\sqrt{n})^3$ converges.

# Motivating Examples

## Cases I and II: Transient and Null Recurrent Irreducible DTMCs

**Observations:**

- Both limits are the same
- Rows in the matrix are identical but add up to 0

# Motivating Examples
## Summary

From the above examples, we can see that

- $\lim_{n\to\infty} P^{(n)}$ doesn't always exist
- $\lim_{n\to\infty} \frac{M^{(n)}}{n+1}$ however always exists and equals $\lim_{n\to\infty} P^{(n)}$ when the later exists. (Why is this intuitively true?)

| Case | $\lim_{n\to\infty} P^{(n)}$ | $\lim_{n\to\infty} \frac{M^{(n)}}{n+1}$ | Identical Rows | Row Sum $= 1$ |
|---|---|---|---|---|
| I | ✓ | ✓ | ✓ | X |
| II | ✓ | ✓ | ✓ | X |
| III | ✓ | ✓ | ✓ | ✓ |
| IV | X | ✓ | ✓ | ✓ |
| V | ✓ | ✓ | X | ✓ |

While the limits for Cases I and II are the same, there is a subtle difference that we will see shortly.

# Motivating Examples

## Summary

The following theorem will help in formally proving the observations made so far.

**Theorem**

*Let $i$ be a recurrent state and $\tilde{m}_i = \mathbb{E}\big[\tilde{T}_i | X_0 = i\big]$ (can be $\infty$).*

1. *If state $i$ is aperiodic,*

$$\lim_{n\to\infty} p_{ii}^{(n)} = 1/\tilde{m}_i$$

2. *If state $i$ is periodic with period $d > 1$,*

$$\lim_{n\to\infty} p_{ii}^{(nd)} = d/\tilde{m}_i$$

The proof is a little involved and follows from another theorem called the Renewal Theorem.

# Limiting Behavior of Irreducible DTMCs

# Limiting Behavior of Irreducible DTMCs

Case I: Transient DTMCs

**Theorem**

*Let* $\{X_n, n \geq 0\}$ *be an transient, irreducible DTMC. Then*

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0 \,\forall\, i,j \in S$$

**Proof.**

Since the DTMC is transient, $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty \,\forall\, i \in S$.Each of the terms in this series is $\geq 0$. Thus, $\lim_{n\to\infty} p_{ii}^{(n)} = 0$.

As the DTMC is irreducible, by definition, $j \to i$ and $\exists\ k \geq 0$ such that $p_{ji}^{(k)} > 0$. Pick an $n \geq k$. Using CK equations,

$$\begin{aligned} p_{jj}^{(n)} &= \sum_{r\in S} p_{jr}^{(k)} p_{rj}^{(n-k)} \\ &\geq p_{ji}^{(k)} p_{ij}^{(n-k)} \end{aligned}$$

Hence, $\lim_{n\to\infty} p_{ij}^{(n-k)} = 0$. ■

# Limiting Behavior of Irreducible DTMCs

Case I: Transient DTMCs

We will see similar results for the null recurrent case. But what's different with transient irreducible DTMCs is that, one can show

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} < \infty \Rightarrow \sum_{n=0}^{\infty} \mathbb{P}\big[X_n \in A\big] < \infty \Rightarrow \mathbb{P}\big[X_n \in A\, i.o.\big] = 0$$

for any finite subset $A \subseteq S$. That is, the DTMC will permanently exit the set $A$ w.p.1.

**Note**: An sequence of events $\{A_n, n \geq 0\}$ is said to occur infinitely often (i.o.) if the event happens for a infinite subsequence of whole numbers.

# Limiting Behavior of Irreducible DTMCs

Case II: Null Recurrent DTMCs

## Theorem

*Let $\{X_n, n \geq 0\}$ be an null recurrent, irreducible DTMC. Then*

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0 \, \forall \, i, j \in S$$

## Proof.

Suppose the DTMC is aperiodic. Since it is null recurrent, from the 'renewal theorem', $\lim_{n\to\infty} p_{ii}^{(n)} = 1/\tilde{m}_i = 0$.

If the DTMC is periodic, again from the 'renewal theorem', $\lim_{n\to\infty} p_{ii}^{(nd)} = d/\tilde{m}_i = 0$. Since it is periodic, $p_{ii}^{d'} = 0$, for all $d'$ which is not an integral multiple of $d$. Thus, $\lim_{n\to\infty} p_{ii}^{(n)} = 0$.

Rest of the proof is similar to the transient case. ∎

# Limiting Behavior of Irreducible DTMCs

## Case II: Null Recurrent DTMCs

For null recurrent DTMCS, unlike the transient case, for any finite $A \subseteq S$,

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} = \infty \Rightarrow \sum_{n=0}^{\infty} \mathbb{P}\big[X_n \in A\big] = \infty \Rightarrow \mathbb{P}\big[X_n \in A \, i.o.\big] = 1$$

Even though $p_{ij}^{(n)} \to 0$!

### Theorem

*Let $e$ be a column vector of ones. For an aperiodic, positive recurrent, irreducible DTMC, there exists unique $\pi_j > 0, j \in S$ such that*

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j, \forall\, i, j \in S$$

$$\pi P = \pi \text{ (Balance Equation)}$$

$$\pi e = 1 \text{ (Normalizing Equation)}$$

### Proof (sketch).

Proof of $\lim_{n\to\infty} p_{ij}^{(n)}$: If $i = j$, 'renewal theorem' is applicable. Suppose $i \neq j$. Consider the pmf of first passage time $\tilde{T}_j$, given $X_0 = i$.

$$u_m = \mathbb{P}\left[\tilde{T}_j = m | X_0 = i\right]$$

What is the support of $\tilde{T}_j$? Thus, $\sum_{m=0}^{\infty} u_m = 1$.

$$p_{ij}^{(n)} = \sum_{m=1}^{n} u_m p_{jj}^{(n-m)}$$

# Limiting Behavior of Irreducible DTMCs

## Case III: Aperiodic, Positive Recurrent DTMCs

### Proof (sketch).

Choose $N$ such that, forall $n \geq N$,

$$\sum_{m=N+1}^{\infty} u_m \leq \epsilon/4 \text{ and } |p_{jj}^{(n)} - \pi_j| \leq \epsilon/2$$

Using, the earlier expression for $p_{ij}^{(n)}$, above inequalities, and triangle inequality, it can be shown that, for $n \geq 2N$,

$$|p_{ij}^{(n)} - \pi_j| \leq \epsilon$$

Proof of $\pi P = \pi$: Using CK equations,

$$p_{jj}^{(n+m)} = \sum_{i \in S} p_{ji}^{(m)} p_{ij}^{(n)}$$

Using bounded convergence theorem and setting $m \to \infty$,

$$\pi_j = \sum_{i \in S} \pi_i p_{ij}^{(n)}$$

# Limiting Behavior of Irreducible DTMCs

Case III: Aperiodic, Positive Recurrent DTMCs

**Proof (sketch).**

Setting $n = 1$ in the above equation gives $\pi P = \pi$. Taking limits $n \to \infty$ yields $\pi e = 1$ as shown below:

$$\pi_j = \sum_{i \in S} \pi_i \lim_{n \to \infty} p_{ij}^{(n)}$$

Establishing uniqueness of $\pi$s is trivial. ∎

Aperiodic, positive recurrent, irreducible DTMCs are called ergodic DTMCs. This theorem is also popularly known as **ergodic theorem**.

The limiting probability distribution is also called **stationary distribution** because if we start with an initial distribution, $a = \pi$, then the pmf of $X_1, X_2, \ldots$ is equal to $\pi$.

Can you draw a connection between $\pi P = \pi$ and eigenvalues of $P$?

We have seen an example which proves that periodic, positive recurrent, irreducible DTMCs do not have a limiting distribution.

But the limiting occupancy distribution can be computed solving the balance and normalizing equations

**Theorem**

*Let $e$ be a column vector of ones. For a periodic, positive recurrent, irreducible DTMC, there exists unique $\pi_j > 0, j \in S$ such that*

$$\lim_{n\to\infty} \frac{m_{ij}^{(n)}}{n+1} = \pi_j, \forall\, i, j \in S$$

$$\pi P = \pi$$

$$\pi e = 1$$

# Limiting Behavior of Irreducible DTMCs

## Necessary and Sufficient Conditions

It turns out that one can check for positive recurrence of irreducible DTMCs using the conditions we just derived.

**Theorem**

*An irreducible DTMC is positive recurrent* $\Leftrightarrow \exists \pi \geq 0$ *that satisfies*

$$\pi P = \pi$$
$$\pi e = 1$$

The interpretation of $\pi$ however depends on whether the DTMC is periodic or a periodic.

# Limiting Behavior of Irreducible DTMCs

Recap

| Case | $\lim_{n\to\infty} P^{(n)}$ | $\lim_{n\to\infty} \frac{M^{(n)}}{n+1}$ | Identical Rows | Row Sum $= 1$ |
|---|---|---|---|---|
| I | ✓ | ✓ | ✓ | X |
| II | ✓ | ✓ | ✓ | X |
| III | ✓ | ✓ | ✓ | ✓ |
| IV | X | ✓ | ✓ | ✓ |

Let's first revisit some examples before studying Case V.

# Lecture Outline

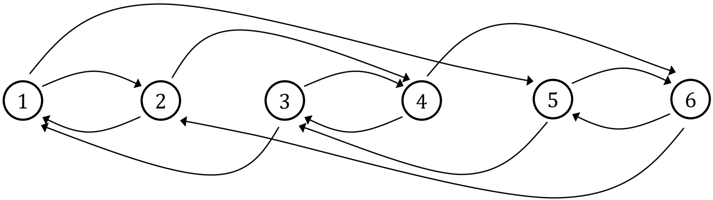
**More Examples**

# More Examples

## Example 1: Shuffling

Recall our card shuffling example from the first class. Which of the five cases does this fall into?

$$\begin{array}{c} \\ \clubsuit\heartsuit\spadesuit \\ \clubsuit\spadesuit\heartsuit \\ \heartsuit\spadesuit\clubsuit \\ \heartsuit\clubsuit\spadesuit \\ \spadesuit\clubsuit\heartsuit \\ \spadesuit\heartsuit\clubsuit \end{array}
\begin{array}{c} \begin{array}{cccccc} \clubsuit\heartsuit\spadesuit & \clubsuit\spadesuit\heartsuit & \heartsuit\spadesuit\clubsuit & \heartsuit\clubsuit\spadesuit & \spadesuit\clubsuit\heartsuit & \spadesuit\heartsuit\clubsuit \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix} \end{array}$$



# More Examples

## Example 1: Shuffling

Solving

$$\pi \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \end{bmatrix} = \pi$$

gives $\pi = [1/6 \; 1/6 \; 1/6 \; 1/6 \; 1/6 \; 1/6]$. A fair shuffling strategy must produce the uniform distribution in the limit and good ones must mix quickly. (The deterministic strategy results in a reducible DTMC.)

**Additional Reading:**


Bayer, D., & Diaconis, P. (1992). Trailing the dovetail shuffle to its lair. The Annals of Applied Probability, 2(2), 294-313.


Addiction to gambling is injurious to health.

Is the page rank example discussed earlier aperiodic, positive recurrent, and irreducible?

$$C = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{array}$$

Solving $\pi P = \pi$, we get $[0.3077 \;\; 0.2308 \;\; 0.1538 \;\; 0.3077]$. Hence, pages 1 and 2 are ranked first, followed by 2 and 3.

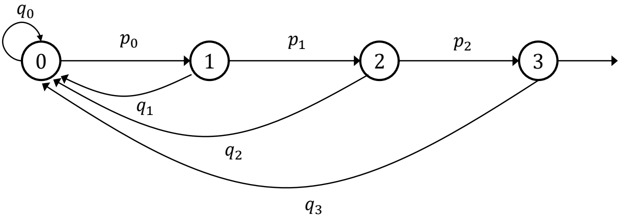
**Additional Reading:**

Brin, S., & Page, L. (1998). The anatomy of a large-scale hypertextual web search engine. Computer networks and ISDN systems, 30(1-7), 107-117.

All the transition probability matrix of the four composers corresponds to Case III. The results of solving $\pi P = \pi$ are shown below:

$$\begin{aligned} \text{Palestrina: } & [0.143\ \ 0.143\ \ 0.143\ \ 0.143\ \ 0.143\ \ 0.143\ \ 0.143] \\ \text{Bach: } & [0.344\ \ 0.089\ \ 0.016\ \ 0.131\ \ 0.273\ \ 0.067\ \ 0.080] \\ \text{Mozart: } & [0.435\ \ 0.086\ \ 0.001\ \ 0.073\ \ 0.330\ \ 0.037\ \ 0.038] \\ \text{Beethoven: } & [0.317\ \ 0.120\ \ 0.012\ \ 0.060\ \ 0.328\ \ 0.034\ \ 0.130] \end{aligned}$$

Consider the general success runs problem. Is it irreducible? Find it's limiting distribution.




Recall that one can directly solve the balance and normalizing equation and if the solution exists, the DTMC is positive recurrent.

# More Examples

## Example 4: Success Runs

The balance equations can be simplified as

$$\pi_{i+1} = p_i \pi_i \,\forall\, i \geq 0$$

Solving this recursively, all $\pi$'s can be written in terms of $\pi_0$ as follows:

$$\pi_i = \rho_i \pi_0 \,\forall\, i \geq 0$$

where $\rho_i = p_0 p_1 \cdots p_{i-1} \,\forall\, i \geq 1$ and $\rho_0 = 1$.

Using the normalizing equation, $\pi_0 = (\sum_{i=0}^{\infty} \rho_i)^{-1}$. Thus, a solution to the above system exists only if $\sum_{i=0}^{\infty} \rho_i < \infty$.

Else, the DTMC is null recurrent or transient and the limiting probabilities are all 0s. It can further be shown from basic definitions that,

- The DTMC is null recurrent if $\sum_{i=0}^{\infty} \rho_i = \infty$ and $\sum_{i=0}^{\infty} q_i = \infty$
- The DTMC is transient if $\sum_{i=0}^{\infty} q_i < \infty$

# Limiting Behavior of Reducible DTMCs

Recall that state spaces of reducible DTMCs can be partitioned as

$$S = C_1 \cup C_2 \cup \ldots \cup C_k \cup \mathcal{C}$$

Let us first renumber states such that $i \in C_r$ and $j \in C_s$ with $r < s$ implies $i < j$. Further, $i \in C_r$ and $j \in \mathcal{C}$ implies $i < j$. Then, the transition matrix can be written in the following format

$$\begin{bmatrix} P(1) & 0 & \ldots & 0 & 0 \\ 0 & P(2) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & P(k) & 0 \\ & D & & & Q \end{bmatrix}$$

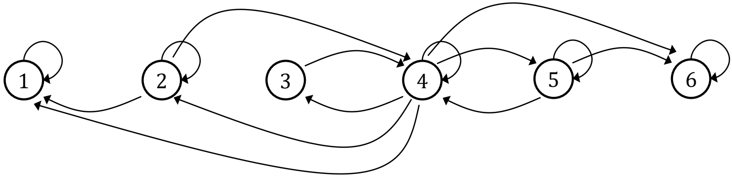
where $P(1), \ldots, P(k)$ are the transition matrices of the $k$ irreducible classes. $Q$ is a $|\mathcal{C}| \times |\mathcal{C}|$ sub-stochastic matrix (row sums are $\leq 1$, why?) and $D$ is a $|\mathcal{C}| \times |S \backslash \{\mathcal{C}\}|$ matrix. We know from earlier analysis limiting distribution of $P(r)^{(n)}$. Since states in $\mathcal{C}$ are transient, one can show that $Q^{(n)} \to 0$.

Thus, the limiting distribution of reducible DTMCs reduces to studying $\lim_{n\to\infty} D^{(n)}$. Let the elements of the $D$ matrix be denoted by $d_{ij}$.

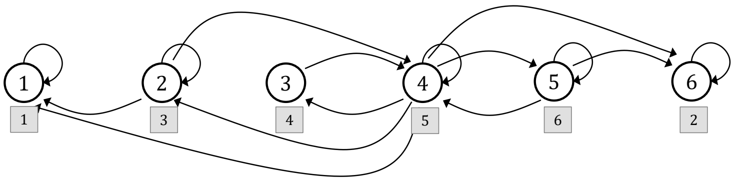
Consider the following DTMC

$$P = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/16 & 1/4 & 1/8 & 1/4 & 1/4 & 1/16 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Partition the state space and convert it to standard form.



States 1 and 6 form closed communicating classes.




The $D$ and $Q$ matrices are shown in blue and green respectively.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 1/2 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/16 & 1/16 & 1/4 & 1/8 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 1/2 \end{bmatrix} \end{array}$$

# Limiting Behavior of Reducible DTMCs

## Key Results

We first need recurrence and transience-style definitions to understand the limiting behavior of $D$. Define,

$$\tilde{T}(r) = \min\{n > 0 : X_n \in C_r\}, 1 \le r \le k$$
$$\tilde{u}_i(r) = \mathbb{P}\big[\tilde{T}(r) < \infty | X_0 = i\big], 1 \le r \le k, i \in \mathcal{C}$$

$\tilde{T}(r)$ is the first passage time and $\tilde{u}_i(r)$ is the absorption probability as it is the probability with which we end up in $C_r$ starting from a state in $\mathcal{C}$.

One can find $\tilde{u}_i$s by solving

$$\tilde{u}_i(r) = \sum_{j \in C_r} p_{ij} + \sum_{j \in \mathcal{C}} p_{ij} \tilde{u}_j(r)$$

# Limiting Behavior of Reducible DTMCs

## Key Results

**Theorem**

*Let $i \in \mathcal{C}$ and $j \in C_r$.*

1. *If $C_r$ is transient or null recurrent $d_{ij}^{(n)} \to 0$*
2. *If $C_r$ is aperiodic and positive recurrent, $d_{ij}^{(n)} \to u_i(r)\pi_j$, where $\pi_j$s are derived from limiting distribution of $P(r)^{(n)}$*
3. *If $C_r$ is periodic and positive recurrent, $d_{ij}^{(n)}$ does not have a limit. However $\sum_{m=0}^{n} d_{ij}^{(m)}/(n+1) \to u_i(r)\pi_j$, where $\pi_j$s are derived from limiting distribution of $P(r)^{(n)}$*

In the example that we discussed earlier, it can be shown that

$$\begin{bmatrix} u_3(1) & u_4(1) & u_5(1) & u_6(1) \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 & 1/2 & 1/4 \end{bmatrix}$$
$$\begin{bmatrix} u_3(2) & u_4(2) & u_5(2) & u_6(2) \end{bmatrix} = \begin{bmatrix} 1/4 & 1/2 & 1/2 & 3/4 \end{bmatrix}$$

Since, $\pi_i = 1$ for each of the two closed communicating classes (states 1 and 2), we can write the limit of $D^{(n)}$ as

$$D^{(n)} \to \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \end{array} \quad P^{(n)} \to \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

# Your Moment of Zen