CE 272
Traffic Network Equilibrium

Lecture 7

System Optimum, Price of Anarchy, and Congestion Pricing

Previously on Traffic Network Equilibrium...

Theorem

$\mathbf{x}^*$ *satisfies the VI,* $\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \, \forall \, \mathbf{x} \in X \Leftrightarrow$ *it satisfies the Wardrop principle*

Previously on Traffic Network Equilibrium...

The User Equilibrium (UE) formulation in terms of the path flows ys is given by

$$\min \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega$$

$$\text{s.t.} \sum_{p \in P_{rs}} y_p = d_{rs} \ \forall\ (r,s) \in Z^2$$

$$y_p \geq 0 \ \forall\ p \in P$$

Previously on Traffic Network Equilibrium...

From the KKT conditions, eliminating λ_p, for all $(r, s) \in Z^2$, $p \in P_{rs}$,

$$\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}(x_{ij}) \geq \mu_{rs}$$

$$y_p \left(\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}(x_{ij}) - \mu_{rs} \right) = 0$$

From the above equations, μ_{rs} is the length of the shortest path.

If $y_p > 0$, then path p must be shortest. If $y_p = 0$, the travel time of path p must be at least μ_{rs}. Voila! Wardrop Principle.

Previously on Traffic Network Equilibrium...

Rewriting the earlier formulation purely in terms of the link flow variables, i.e., the decision variables are the xs.

$$\min \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in A} x_{ij}^{rs} - \sum_{h:(h,i)\in A} x_{hi}^{rs} = \begin{cases} d_{is} & \text{if } i = r \\ -d_{ri} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \quad \forall\, (r,s) \in Z^2$$

$$x_{ij} = \sum_{(r,s)\in Z^2} x_{ij}^{rs} \,\forall\, (i,j) \in A$$

$$x_{ij}^{rs} \geq 0 \,\forall\, (i,j) \in A, (r,s) \in Z^2$$

This optimization program, also called the Beckmann formulation, has fewer variables and is easier to solve.

Previously on Traffic Network Equilibrium...

The objective is again convex in the aggregate link flows if the delay functions are non-decreasing. (Why?)

$$\min \sum_{(i,j) \in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

In addition, if we assume that the delay functions are strictly increasing, the objective is strictly convex. (Why?)

Thus, **for strictly increasing delay functions, the equilibrium aggregate link flows are unique but the path flows need not be.**

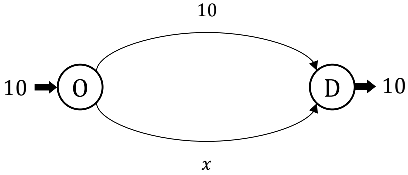
Lecture Outline

1. System Optimum
2. Price of Anarchy
3. Congestion Pricing

System Optimum

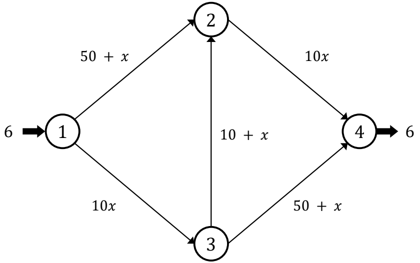
System Optimum
Introduction

The User Equilibrium (UE) state is realized when travelers behave selfishly. We saw an example in which this is not optimal for the overall network.



The TSTT for the UE flows is 100 but it is possible to split travelers evenly and reduce it to 75.

The Braess network also exhibits a similar property.



The UE solution has 2 travelers on each of the three routes with a TSTT of $6(92) = 552$. However, if we route 3 travelers each along 1-2-4 and 1-3-4, the TSTT is $6(83) = 498$.

System Optimum

Introduction

Uncoordinated routing can be detrimental since it results in more congestion, fuel usage, and pollution.

On the other hand, flows that minimize TSTT are called system optimum (SO) or social optimum but they are not self-enforcing since someone might have an incentive to deviate.

They can be achieved if the routing decisions are made by a centralized mechanism. But can this happen? (Are there any other options?)

The SO problem can also be formulated as an optimization program. The flow conservation constraints remain the same as before.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) \\
\text{s.t.} \quad & \sum_{p\in P_{rs}} y_p = d_{rs} \; \forall \; (r,s) \in Z^2 \\
& x_{ij} = \sum_{p\in P} \delta_{ij}^p y_p \, \forall \, (i,j) \in A \\
& y_p \geq 0 \; \forall \; p \in P
\end{aligned}
$$

System Optimum

Uniqueness

Is the objective $\sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij})$ convex? Strictly convex?

Lecture Outline

Price of Anarchy

Price of Anarchy

Introduction

In the two-link network, the TSTT improved to 75 from 100, i.e., the UE solution is 33.34% worse than that of the SO solution.

In the case of the Braess network we noticed an improvement from 552 to 498, i.e., the UE solution is 10.84% worse than the SO.

How bad is selfish routing?

Price of Anarchy

Introduction

The difference in TSTT appears to depend on

1. Demand
2. Network topology
3. Delay functions

Can we find a worst case scenario? In other words, can we bound the inefficiency due to selfish routing and claim that the TSTT of the UE flows is at most x% worse than the TSTT of the SO solution?

Tim Roughgarden addressed this question in 2004 in his seminal dissertation *Selfish routing and the price of anarchy* [PDF].

Price of Anarchy

Results

The Price of Anarchy (PoA) for the traffic assignment problem is defined as

$$PoA = \frac{TSTT_{UE}}{TSTT_{SO}}$$

Proposition

For linear, non-negative, and non-decreasing delay functions, the PoA is at most 4/3

Notice that with mild assumptions on the delay functions, we are guaranteed that the UE solution cannot be worse than the SO flows by 33.34% irrespective of the network topology and demand!

Roughgarden found similar bounds for other types of polynomial delay functions.

Description	Representative	PoA bound
Linear	$ax + b$	$\frac{4}{3} \approx 1.333$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.626$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.896$
Degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} = \Theta(\frac{p}{\ln p})$

It was also shown that the worst case PoA occurs in the two-link network shown earlier for all the above types of delay functions.

Price of Anarchy

Linear Delay Functions

We will use a VI approach to prove the PoA result for linear delay functions. A second (longer) method can be found in Roughgarden's dissertation.

Proposition

For linear, non-negative, and non-decreasing delay functions, the PoA is at most 4/3

Proof.

From the VI definition of UE,

$$\mathbf{t}(\mathbf{x}^{UE})^T(\mathbf{x} - \mathbf{x}^{UE}) \geq 0 \, \forall \, \mathbf{x} \in X$$

Therefore, for any feasible $\mathbf{x}$,

$$\begin{aligned} TSTT_{UE} &\leq \mathbf{t}(\mathbf{x}^{UE})^T \mathbf{x} \\ &= \mathbf{t}(\mathbf{x}^{UE})^T \mathbf{x} + \mathbf{t}(\mathbf{x})^T \mathbf{x} - \mathbf{t}(\mathbf{x})^T \mathbf{x} \\ &= [\mathbf{t}(\mathbf{x}^{UE}) - \mathbf{t}(\mathbf{x})]^T \mathbf{x} + \mathbf{t}(\mathbf{x})^T \mathbf{x} \end{aligned}$$

Proof.

Let the delay functions be $t_{ij}(x_{ij}) = a_{ij}x_{ij} + b_{ij}$. Consider the expression $[\mathbf{t}(\mathbf{x}^{UE}) - \mathbf{t}(\mathbf{x})]^T\mathbf{x}$. For any feasible $\mathbf{x}$,

$$\begin{aligned}
[\mathbf{t}(\mathbf{x}^{UE}) - \mathbf{t}(\mathbf{x})]^T\mathbf{x} &= \sum_{(i,j)\in A} \Big(x_{ij}t_{ij}(x_{ij}^{UE}) - x_{ij}t_{ij}(x_{ij}) \Big) \\
&= \sum_{(i,j)\in A} x_{ij}(a_{ij}x_{ij}^{UE} + b_{ij}) - \sum_{(i,j)\in A} x_{ij}(a_{ij}x_{ij} + b_{ij})
\end{aligned}$$

Notice that $x_{ij}x_{ij}^{UE} \leq x_{ij}^2 + \frac{(x_{ij}^{UE})^2}{4}$ (Why?). Hence,

$$\begin{aligned}
[\mathbf{t}(\mathbf{x}^{UE}) - \mathbf{t}(\mathbf{x})]^T\mathbf{x} &\leq \frac{1}{4}\sum_{(i,j)\in A} x_{ij}^{UE}(a_{ij}x_{ij}^{UE}) \\
&\leq \frac{1}{4}\sum_{(i,j)\in A} x_{ij}^{UE}(a_{ij}x_{ij}^{UE} + b_{ij}) = \frac{1}{4} TSTT_{UE}
\end{aligned}$$

Price of Anarchy

Linear Delay Functions

Proof.

$$
\begin{aligned}
TSTT_{UE} &\leq [\mathbf{t}(\mathbf{x}^{UE}) - \mathbf{t}(\mathbf{x})]^T \mathbf{x} + \mathbf{t}(\mathbf{x})^T \mathbf{x} \\
&\leq \frac{1}{4} TSTT_{UE} + \mathbf{t}(\mathbf{x})^T \mathbf{x}
\end{aligned}
$$

Since the above expression is true for all feasible $\mathbf{x}$, it holds for $\mathbf{x}^{SO}$. Therefore,

$$
\begin{aligned}
TSTT_{UE} &\leq \frac{1}{4} TSTT_{UE} + TSTT_{SO} \\
\Rightarrow \frac{TSTT_{UE}}{TSTT_{SO}} &\leq \frac{4}{3}
\end{aligned}
$$

∎

Price of Anarchy

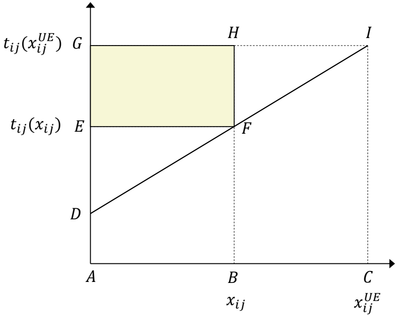
Linear Delay Functions

One can also use a graphical approach to prove

$$[\mathbf{t}(\mathbf{x}^{UE}) - \mathbf{t}(\mathbf{x})]^T \mathbf{x} \leq \frac{1}{4} TSTT_{UE}$$

As seen earlier, the above inequality holds component-wise, i.e.,

$$x_{ij} t_{ij}(x_{ij}^{UE}) - x_{ij} t_{ij}(x_{ij}) \leq \frac{1}{4} x_{ij}^{UE} t_{ij}(x_{ij}^{UE})$$



Case I: $x_{ij} \geq x_{ij}^{UE}$
Trivial.

Case II: $x_{ij} < x_{ij}^{UE}$
Area of EFHG $\leq \frac{1}{2}$ Area of DIG
Area of DIG $\leq \frac{1}{2}$ Area of ACIG

Congestion Pricing

Congestion Pricing

Marginal Costs

The UE and SO problems have the same constraints but differ in the objectives.

UE Objective:

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

SO Objective:

$$\sum_{(i,j)\in A} x_{ij}\, t_{ij}(x_{ij})$$

Is there a way to get one from the other using some transformation?

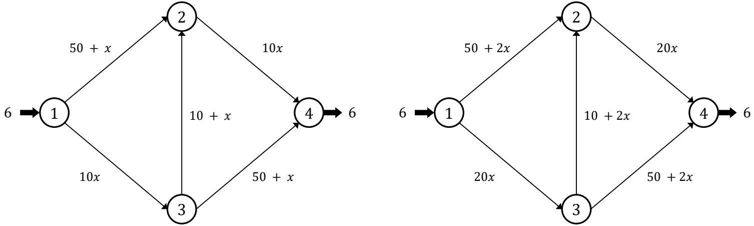
Congestion Pricing

Marginal Costs

Replacing the delay functions with $\hat{t}_{ij}(x) = t_{ij}(x) + x\, t'_{ij}(x)$ and solving the UE problem gives us the SO solution!

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega) + \omega\, t'_{ij}(\omega)\, d\omega = \sum_{(i,j)\in A} \int_0^{x_{ij}} d(\omega\, t_{ij}(\omega))$$

$$= \sum_{(i,j)\in A} x_{ij}\, t_{ij}(x_{ij})$$



Congestion Pricing
Marginal Costs

The function $\hat{t}_{ij}(x) = t_{ij}(x) + x\, t'_{ij}(x)$ is called the marginal cost function. It consists of

- The original delay function $t_{ij}(x)$
- The externality caused by an additional traveler $xt'_{ij}(x)$

Externalities are costs/benefits incurred due to one's actions by all the other agents in the system.

In the context of traffic, when an additional traveler takes link (i, j) he or she increases the travel time by $t'_{ij}(x)$. This imposes a negative externality of $xt'_{ij}(x)$ on all users on (i, j).

Proposition (System Optimal)

At an SO state, all used routes have equal and minimal marginal costs.

Thus, by setting tolls that equal the congestion externalities imposed by a traveler one can achieve a SO solution! In other words, solve the SO problem and set a toll of $x_{ij}^{SO} t'_{ij}(x_{ij}^{SO})$ on each link.

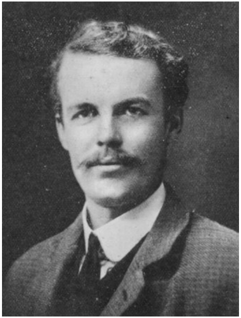
When a network has tolls, we will assume that travelers minimize **generalized cost of travel** $= \gamma$ (travel time) $+$ toll.

γ is the value of time (VoT) measured in ₹/min. For now, assume that all travelers have the same VoT of 1.

Congestion Pricing

Marginal Costs

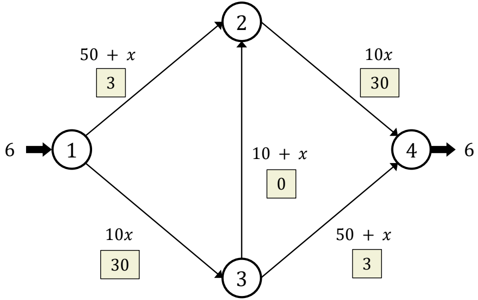
The idea of externalities and tolls is very old. It was first discussed by Arthur Pigou in his classic book *The Economics of Welfare* in 1920 (over a 100 years ago!)



Congestion Pricing

Example

What is the UE solution in the following network with tolls?

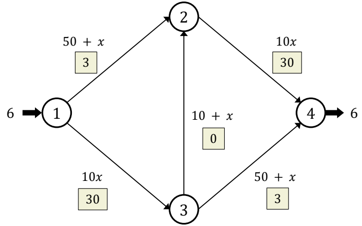


Congestion Pricing

Minimum Revenue Tolls

First-best tolls for achieving SO flows are not unique. Hence, one can seek tolls that satisfy some secondary objective.

What are the UE flows in the following networks?



What is the total revenue in both cases? Why might we want to collect minimum revenue tolls?

Congestion Pricing

Other Objectives

By congestion pricing, we typically refer to tolling mechanisms which minimize or eliminate inefficiency caused due to selfish behavior.

In practice, tolls can be used for a variety of other purposes:

- Revenue maximization
- Maintain a certain level of service on managed lanes
- Financing mechanism for BOT projects

Congestion Pricing

Types of Tolls

The congestion pricing mechanism discussed earlier assumes that all links can be tolled (First-best pricing).

However, this is expensive and difficult to execute and practical implementations often fall under the following categories.

- Facility-based schemes
- Cordon pricing (Stockholm)
- Zonal schemes (London)

Models in which a subset of links are tolled are called second-best pricing schemes. Since it is a constrained version we are not guaranteed to achieve an SO.

These models can be viewed as Stackelberg games in which the leader is the toll operator and the travelers are the followers.

Optimal tolls in such settings are harder to compute and the following methods are typically used.

- Bilevel Programs
- Mathematical Programs with Equilibrium Constraints (MPECs)

Congestion Pricing

Bilevel Programs for Second-best Pricing

Bilevel programs have two optimization models: one at the upper level and another at the lower level.

For instance, at the upper level, the objective could be

$$\min_{\mathbf{c}} \sum_{(i,j)\in A} x_{ij}(\mathbf{c}) t_{ij}(x_{ij}(\mathbf{c}))$$
$$\text{s.t. } c_{ij} = 0 \, \forall \, (i,j) \in A'$$

where A' is the set of links on which tolls cannot be collected and $x_{ij}(\mathbf{c})$s are the solution to the lower level UE problem

$$\min_{\mathbf{x}\in X} \sum_{(i,j)\in A} \int_0^{x_{ij}} \left(t_{ij}(\omega) + c_{ij}\right) \, d\omega$$

Note that the c values are assumed to be fixed in the lower level.

MPECs on the other hand model the Wardrop equilibrium conditions as constraints.

These equilibrium constraints are either in the from of VIs or KKT conditions.

However, with these equilibrium constraints, Slater's conditions do not hold and additional assumptions are needed to invoke other types of constraint qualifications and to derive the necessary conditions for optimality.

Suppose $\tau_p(\mathbf{y})$ denotes the travel time on path p given a path flow vector $\mathbf{y}$. Let the feasible region of path flows be represented as

$$Y = \left\{ \mathbf{y} : \sum_{p \in P_{rs}} y_p = d_{rs} \, \forall \, (r,s) \in Z^2, y_p \geq 0 \, \forall \, p \in P \right\}$$

The second-best tolling problem with VI-based equilibrium constraints can be written as

$$\begin{gathered} \min_{\mathbf{y},\mathbf{c}} \sum_{p \in P} y_p \tau_p(\mathbf{y}) \\ \text{s.t. } \mathbf{y} \in Y \\ c_{ij} = 0 \, \forall \, (i,j) \in A' \\ [\boldsymbol{\tau}(\mathbf{y}) + \Delta^T \mathbf{c}]^T (\mathbf{y}' - \mathbf{y}) \geq 0 \, \forall \, \mathbf{y}' \in Y \end{gathered}$$

Note that the tolls do not feature in the objective since they are **transfer payments**. We assume that they are returned to the system and hence it does not matter how much toll is collected.

The issue with this formulation is that it has infinitely many constraints (one for each feasible $\mathbf{y}'$). Alternately, we can restrict our attention to the extreme points of Y. Suppose $\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^l$ are the extreme points of Y.

$$\min_{\mathbf{y},\mathbf{c}} \sum_{p \in P} y_p \tau_p(\mathbf{y})$$

$$\text{s.t. } \mathbf{y} \in Y$$

$$c_{ij} = 0 \, \forall \, (i,j) \in A'$$

$$[\boldsymbol{\tau}(\mathbf{y}) + \Delta^T \mathbf{c}]^T (\mathbf{y}^i - \mathbf{y}) \geq 0 \, \forall i = 1, 2, \ldots, l$$

The number of extreme points, though finite, are exponential and hence these problems are typically solved using a column generation-type method.

If the link delay functions are separable and non-decreasing, the VIs can be replaced with the KKT conditions of the Beckmann formulation.

$$\min_{\mathbf{y},\mathbf{c},\boldsymbol{\mu}} \sum_{p\in P} y_p \tau_p(\mathbf{y})$$

$$\text{s.t. } \mathbf{y} \in Y$$

$$c_{ij} = 0 \,\forall\, (i,j) \in A'$$

$$\tau_p(\mathbf{y}) + \sum_{(i,j)\in A} \delta_{ij}^p c_{ij} \geq \mu_{rs} \,\forall\, (r,s) \in Z^2, p \in P_{rs}$$

$$y_p\left(\tau_p(\mathbf{y}) + \sum_{(i,j)\in A} \delta_{ij}^p c_{ij} - \mu_{rs}\right) = 0 \,\forall\, (r,s) \in Z^2, p \in P_{rs}$$

Your Moment of Zen