CE 272
Traffic Network Equilibrium

Lecture 6

Wardrop User Equilibrium and Beckmann Formulation

Previously on Traffic Network Equilibrium...

When does strong duality hold? In many cases, some of which do not even require convexity! The conditions (called constraint qualifications) however are usually complicated and we do not need to know much about it for this course.

Let's look at one instance called **Slater's condition**. If our primal was of the form

$$\begin{aligned} \min_{\mathbf{x}} \ & f(\mathbf{x}) \\ \text{s.t.} \ & g_i(\mathbf{x}) \leq 0 \ \ \forall\, i = 1, 2, \ldots, l \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

where f and gs are all convex and there exist a feasible $\mathbf{x}$ such that $g_i(\mathbf{x}) < 0 \,\forall\, i = 1, \ldots l$, then strong duality holds.

Previously on Traffic Network Equilibrium...

Proposition (Necessary KKT Conditions)

Assuming strong duality holds, any $\mathbf{x}^*$ and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ that are optimal for the primal and dual problems must satisfy

- *Primal Feasibility*

$$g_i(\mathbf{x}^*) \leq 0 \,\forall\, i = 1, \ldots, l$$
$$h_i(\mathbf{x}^*) = 0 \,\forall\, i = 1, \ldots, m$$

- *Dual Feasibility*

$$\boldsymbol{\lambda}^* \geq \mathbf{0}$$

- *Complementary Slackness*

$$\lambda_i^* g_i(\mathbf{x}^*) = 0 \,\forall\, i = 1, \ldots, m$$

- *Gradient of the Lagrangian vanishes*

$$\nabla_{\mathbf{x}} f(\mathbf{x}^*) + \sum_{i=1}^{l} \lambda_i^* \nabla_{\mathbf{x}} g_i(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i^* \nabla_{\mathbf{x}} h_i(\mathbf{x}^*) = \mathbf{0}$$

Previously on Traffic Network Equilibrium...

Proposition (Sufficient KKT Conditions)

Suppose f, g_i, and h_i are all differentiable and convex. Then, any $\bar{\mathbf{x}}$ and $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ that satisfy the following KKT conditions are optimal to the primal and dual and the duality gap is 0.

$$g_i(\bar{\mathbf{x}}) \leq 0 \, \forall \, i = 1, \ldots, l$$

$$h_i(\bar{\mathbf{x}}) = 0 \, \forall \, i = 1, \ldots, m$$

$$\bar{\boldsymbol{\lambda}} \geq \mathbf{0}$$

$$\bar{\lambda}_i g_i(\bar{\mathbf{x}}) = 0 \, \forall \, i = 1, \ldots, m$$

$$\nabla_{\mathbf{x}} f(\bar{\mathbf{x}}) + \sum_{i=1}^{l} \bar{\lambda}_i \nabla_{\mathbf{x}} g_i(\bar{\mathbf{x}}) + \sum_{i=1}^{m} \bar{\mu}_i \nabla_{\mathbf{x}} h_i(\bar{\mathbf{x}}) = \mathbf{0}$$

Previously on Traffic Network Equilibrium...

LP formulation for shortest paths:

$$\min \sum_{(i,j)\in A} t_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in A} x_{ij} - \sum_{h:(h,i)\in A} x_{hi} = \begin{cases} 1 & \text{if } i = r \\ -1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} \geq 0 \, \forall \, (i,j) \in A$$

Equality constraints can be written as $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is an $n \times m$ matrix, $\mathbf{x}$ is a $m \times 1$ vector, and $\mathbf{b}$ is a $n \times 1$ vector.

Previously on Traffic Network Equilibrium...

Primal feasibility:

$$\mathbf{Ax} = \mathbf{b}$$
$$0 \leq x_{ij} \leq 1 \, \forall \, (i, j) \in A$$

Dual feasibility:

$$\lambda_{ij} \geq 0 \, \forall \, (i, j) \in A$$

Complementary Slackness:

$$\lambda_{ij} x_{ij} = 0 \, \forall \, (i, j) \in A$$

Gradient of the Lagrangian vanishes:

$$\lambda_{ij} = t_{ij} + \mu_i - \mu_j \, \forall, (i, j) \in A$$

From the above conditions, interpreting $\boldsymbol{\mu}$s as the distance labels, we get the Bellman's conditions: $\mu_j \leq t_{ij} + \mu_i$ and if $x_{ij} = 1 \Rightarrow \mu_j = \mu_i + t_{ij}$

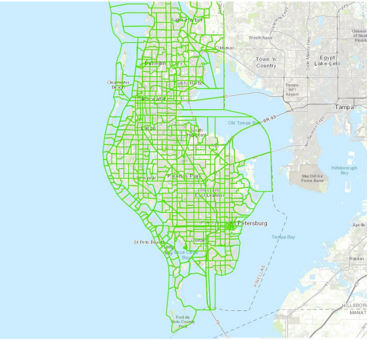
Lecture Outline

1. Preliminaries
2. Path-based Formulation
3. Link-based Formulation

Lecture Outline

Preliminaries

First, let's extend our definition of a graph to include a subset of nodes from which trips originate or end. These nodes are called **zone centroids** and can be actual junctions or artificial nodes.



If zone centroids are artificially created, they are connected to nearby streets using artificial links called **centroid connectors**.

It is assumed that artificially created centroid connectors can be traversed instantaneously.

Preliminaries

Demand

The demand information for all OD pairs is commonly referred to as **OD matrix** or **trip tables**.

The number of person trips are computed from the first two steps of the four-step process. In the third step, these trips are assigned to different modes (car, bus, two-wheeler etc.) resulting in a trip table for each mode.

But for equilibrium analysis, we assume that demand comprises of only passenger cars.* The demand of other types of vehicles are adjusted by factors called **passenger car units (PCUs)** that reflect their sizes relative to that of a car.

- Bicycle: 0.2
- Motorcycle: 0.5
- Buse: 3.5

* This assumption will be relaxed later.

Preliminaries

BPR Functions

The travel time on a link will be assumed to be purely a function of the flow on it (**Separability assumption**).

Typically, link travel times are assumed to follow the Bureau of Public Roads (BPR) function

$$t_{ij}(x_{ij}) = t_{ij}^0 \left(1 + \alpha \left(\frac{x_{ij}}{C_{ij}}\right)^{\beta}\right)$$

t_{ij}^0 is the free-flow travel time, x_{ij} is the flow on link (i, j), and C_{ij} is the capacity or throughput, which is the maximum number of vehicles that can pass through a cross section of the road (both are measured in *vehicles/hr*).

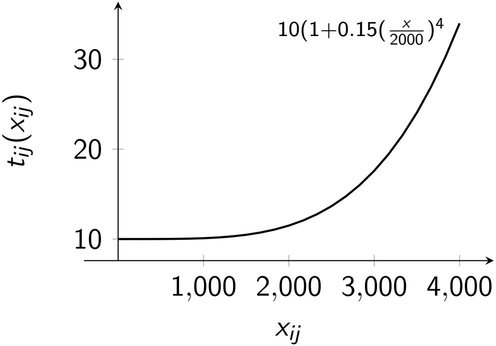
Link travel time functions also called **link-performance functions, delay functions, or latency functions**.

* The α and β in the above expression are the same as `B` and `Power` in your Programming Task 1

Preliminaries

BPR Functions

Commonly used parameter values in the BPR functions are $\alpha = 0.15$ and $\beta = 4$. One can calibrate these values for different links using actual traffic counts.

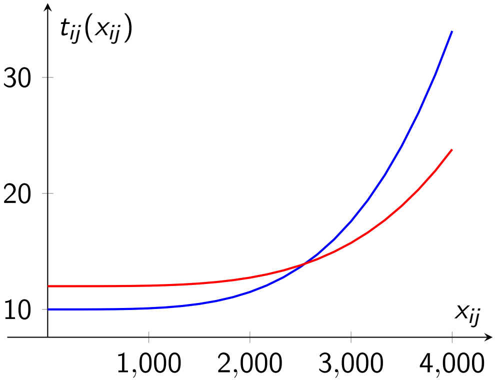


Note that the BPR functions are non-negative and strictly increasing.

Preliminaries

BPR Functions

What are the equilibrium flows in a network of two parallel links (red and blue) with the following delay functions when the total demand is (a) 2000 and (b) 5000.



Preliminaries

BPR Functions

- The BPR functions are defined for flow values that exceed capacity. We'll ignore capacity constraints for now and look at formulations and solution techniques which explicitly model capacity constraints later.

- Alternately, we can define delay functions that exhibit a steep increase at flows close to the roadway capacity.

- Practitioners often use V/C ratios, i.e., x_{ij}/C_{ij}, to identify the links that are heavily congested.

Preliminaries

Link and Path Flows

Equilbrium solutions can be computed in terms of the link flows or the path flows.

Knowledge of either of them lets us compute link travel times using the delay functions.

The travel time on a path is simply the sum of the travel times on the links belonging to the path.

Preliminaries

Link and Path Flows

We will denote the link flow vector using $\mathbf{x}$ and the set of feasible link flows (ones that satisfy flow conservation) as X.

Path flows are denoted by $\mathbf{y}$ and the set of feasible path flows are represented using Y. (Should we include paths with cycles?)

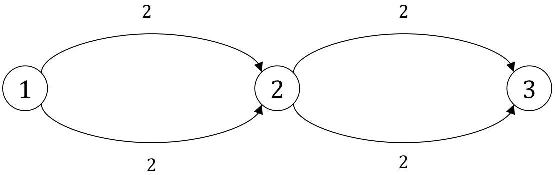
Given a path flow vector $\mathbf{y}$, the link flows uniquely $\mathbf{x}$. Let δ_{ij}^p denote a indicator variable which is 1 if link (i, j) is in path p and is 0 otherwise.

Define a matrix of δs called a link-path incidence matrix Δ in which rows represent links and columns represent paths.

$$\mathbf{x} = \Delta \mathbf{y}$$

Preliminaries
Link and Path Flows

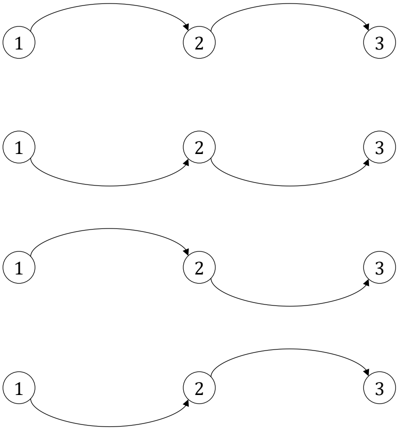
Given a link flow vector **x**, the path flows cannot however be uniquely identified.



Can you find multiple path flows in the above network?

Preliminaries

Link and Path Flows



Path Flow 1	Path Flow 2	Path Flow 3
1	2	0
1	2	0
1	0	2
1	0	2

Path-based Formulation

Path-based Formulation

We will start with a path-based formulation to establish connections with the Wardrop principle.

However, from a computational standpoint, this formulation is not an ideal choice since the number of paths can be very large.

Denote the set of paths between an OD pair (r, s) as P_{rs}. Let the set of all paths between all OD pairs be $P = \cup_{(r,s) \in Z^2} P_{rs}$.

Path-based Formulation

Optimization Model

To define an optimization model, we need

- Objective
- Decision Variables
- Constraints

The decision variables are the path flows and the constraints are the flow conservation constraints.

But how do we define the objective so that the optimal values satisfy Wardrop equilibria?

Path-based Formulation

Optimization Model

Can we reverse engineer a convex function such that the KKT conditions are equivalent to the Wardrop equilibria?

Martin Beckmann, C. B. McGuire, and Christopher Winsten in 1956 discovered such a function in their seminal book *Studies in the Economics of Transportation*.

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

This function is commonly referred to as the Beckmann function.

Since the decision variables are path flows, we will replace x_{ij} in the objective $\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$ with $\sum_{p\in P} \delta_{ij}^p y_p$.

The complete formulation purely in terms of the ys take the form

$$\min \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega$$

$$\text{s.t.} \quad \sum_{p\in P_{rs}} y_p = d_{rs} \ \forall\ (r,s) \in Z^2$$

$$y_p \geq 0 \ \forall\ p \in P$$

Path-based Formulation

Optimization Model

- If the delay functions are non-decreasing, is the objective convex?
- Does Slater's condition hold?

$$
\begin{aligned}
\min \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega \\
\text{s.t.} \quad \sum_{p\in P_{rs}} y_p = d_{rs} \; \forall \; (r,s) \in Z^2 \\
y_p \geq 0 \; \forall \; p \in P
\end{aligned}
$$

What are the KKT conditions for the above formulation?

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega + \sum_{p\in P} \lambda_p(-y_p) + \sum_{(r,s)\in Z^2} \mu_{rs} \left(d_{rs} - \sum_{p\in P_{rs}} y_p \right)$$

Path-based Formulation

KKT Conditions

Primal feasibility:

$$\sum_{p \in P_{rs}} y_p = d_{rs} \; \forall \; (r, s) \in Z^2$$

$$y_p \geq 0 \; \forall \; p \in P$$

Dual feasibility:

$$\lambda_p \geq 0 \, \forall \, p \in P$$

Complementary Slackness:

$$\lambda_p y_p = 0 \, \forall \, p \in P$$

Gradient of the Lagrangian vanishes:

$$\sum_{(i,j) \in A} \delta_{ij}^p t_{ij}(x_{ij}) - \lambda_p - \mu_{rs} = 0 \, \forall \, (r, s) \in Z^2, p \in P_{rs}$$

Path-based Formulation

KKT Conditions

From the last three conditions, eliminating λ_p, for all $(r,s) \in Z^2$, $p \in P_{rs}$,

$$\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}(x_{ij}) \geq \mu_{rs}$$

$$y_p \left(\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}(x_{ij}) - \mu_{rs} \right) = 0$$

From the above equations, μ_{rs} is the length of the shortest path.

If $y_p > 0$, then path p must be shortest. If $y_p = 0$, the travel time of path p must be at least μ_{rs}. Voila! Wardrop Principle.

Path-based Formulation

Uniqueness

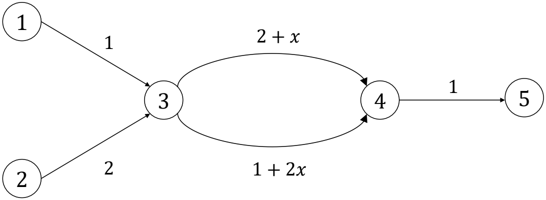
Assuming that the travel times are **non-decreasing**, we showed that the objective is **convex**.

Does the above formulation have a unique optimum?

Path-based Formulation

Uniqueness

The objective is convex, but it need not be strictly convex. Hence, the solution to the optimization model need not be unique.

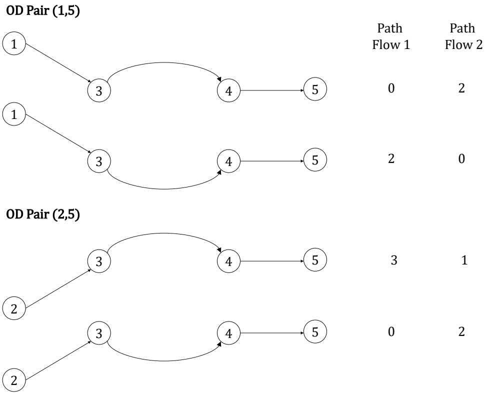


For example, in the above network, assume there are 2 travelers from 1 to 5 and 3 travelers from 2 to 5.

Path-based Formulation

Uniqueness

There are multiple path flow solutions which satisfy the Wardrop priciple.



For both solutions, the travel time on paths between OD pair (1,5) is 7 and the travel times on paths between (2,5) is 8.

Lecture Outline

Link-based formulation

Link-based Formulation

Change of Variables

We can rewrite the earlier formulation purely in terms of the link flow variables, i.e., the decision variables are the xs.

$$\min \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in A} x_{ij}^{rs} - \sum_{h:(h,i)\in A} x_{hi}^{rs} = \begin{cases} d_{is} & \text{if } i = r \\ -d_{ri} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \quad \forall\, (r,s) \in Z^2$$

$$x_{ij} = \sum_{(r,s)\in Z^2} x_{ij}^{rs}\, \forall\, (i,j) \in A$$

$$x_{ij}^{rs} \geq 0\, \forall\, (i,j) \in A, (r,s) \in Z^2$$

This optimization program, also called the Beckmann formulation, has fewer variables and is easier to solve.

Link-based Formulation

Uniqueness

The objective is again convex if the delay functions are non-decreasing. (Why?)

$$\min \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

In addition, if we assume that the delay functions are strictly increasing, the objective is strictly convex. (Why?)

Thus, **for strictly increasing delay functions, the equilibrium link flows are unique but the path flows need not be.**

At the start of this lecture, we wanted an objective that represented the 'energy' of the system. The Beckmann function essentially serves this purpose but it does not have any physical meaning or interpretation.

It is also called the potential function and we will learn more about such functions soon.

In the next few lectures, we will explore methods to solve the link-based formulation.

Your Moment of Zen