CE 272
Traffic Network Equilibrium

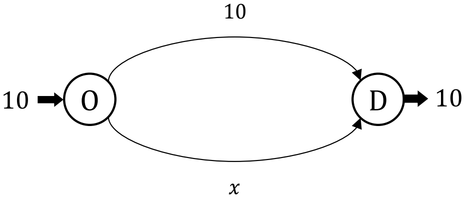
Lecture 4

Fixed Points and Variational Inequalities

Previously on Traffic Network Equilibrium...

Nash Equilibrium (1951)

At equilibrium, no player has an incentive to deviate.

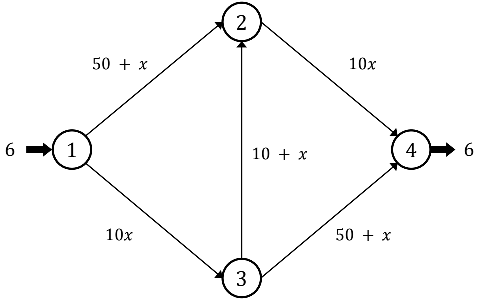


Wardrop Equilibrium (1952)

All used paths have equal and minimal travel time.

Previously on Traffic Network Equilibrium...

Braess Network



Previously on Traffic Network Equilibrium...

Proposition (Necessary and Sufficient Conditions)

$\mathbf{x}^*$ *is a global minimum of a differentiable convex function* $f : X \subseteq \mathbb{R}^n \to \mathbb{R} \Leftrightarrow \nabla f(\mathbf{x}^*) = \mathbf{0}$

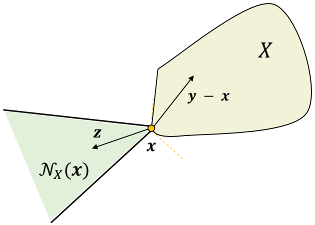
Previously on Traffic Network Equilibrium...

Definition (Normal Cone)

Let $X \subseteq \mathbb{R}^n$, the *normal cone* of X at $\mathbf{x}$ is defined as

$$\mathcal{N}_X(\mathbf{x}) = \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z}^T(\mathbf{y} - \mathbf{x}) \leq 0, \ \forall \ \mathbf{y} \in X\}$$

For the purpose of the following illustration, assume $\mathbf{x}$ is the origin.



Proposition (Necessary and Sufficient Conditions)

$\mathbf{x}^*$ *is an optimal to the convex program* $\min f(\mathbf{x})$ *s.t.* $\mathbf{x} \in \mathcal{X}$ *iff* $-\nabla f(\mathbf{x}^*) \in \mathcal{N}_{\mathcal{X}}(\mathbf{x}^*)$

Lecture Outline

- Visualizing Equilibria
- Fixed Points
- Variational Inequalities

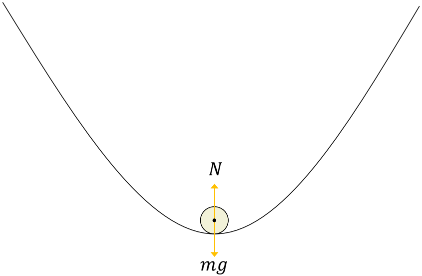
Lecture Outline

Visualizing Equilibria

Visualizing Equilibria

Statics

In the very first lecture we saw examples of equilibrium flows in traffic networks. Are these ideas related to the notion of equilibrium in other fields?

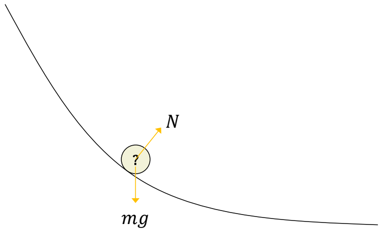


Consider the ball in the above figure. If left undisturbed, it will remain stationary because the normal force and gravitational force balance each other. In other words, it is in a state of **equilibrium**.

Visualizing Equilibria

Statics

What happens in this scenario?

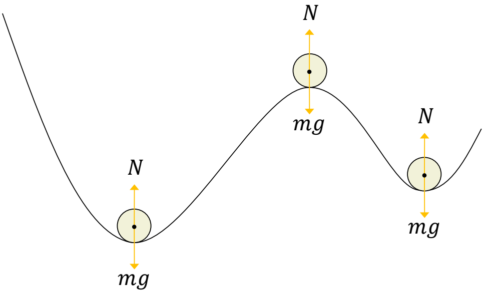


Equilibrium does not exist!

Visualizing Equilibria

Statics

How about this one?

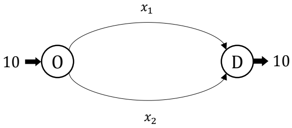


Equilibrium may not be unique!

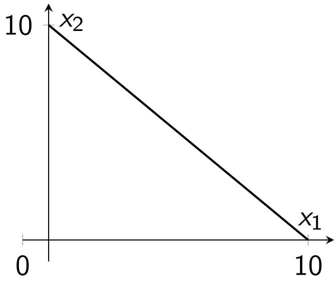
Visualizing Equilibria

Two-path Network

Consider the two-path network in the adjacent figure. The feasible flows must satisfy $x_1 + x_2 = 10$, where x_1 and x_2 are the flows on the top and bottom paths.



The feasible region can be represented using the following line segment.



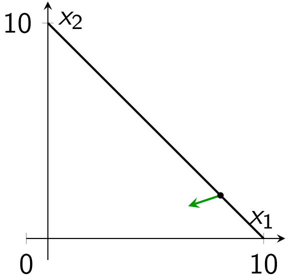
Visualizing Equilibria

Two-path Network

Let the travel times on the two links depend on the flow vector $\mathbf{x} = (x_1, x_2)$ and be denoted as $\mathbf{t}(\mathbf{x}) = (t_1(\mathbf{x}), t_2(\mathbf{x}))$.

Higher the path travel time, greater is the chance that path flow **reduces** since users will shift to other paths. Hence, let's imagine that at every point in the feasible region, a force field (or payoffs) $-\mathbf{t}(\mathbf{x})$ exists.

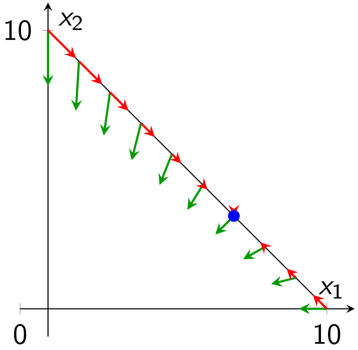
If an object placed at a point in the feasible region does not move (within the feasible region) under this force field, we say it is at equilibrium. We'll later formally show that such points are in fact Wardrop equilibria.



Visualizing Equilibria

Two-path Network

Alternately, we can project the force vector on the feasible region to represent components of the force field that can cause any displacement (also called projected payoffs).

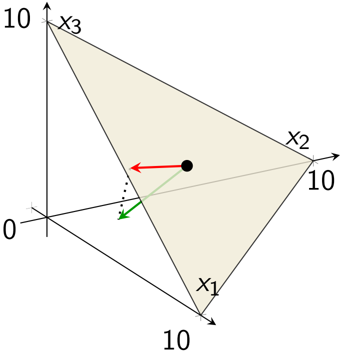


The projected payoffs can also be used to identify equlibria.

Visualizing Equilibria

Three-path Network

Now suppose there are three parallel paths between O and D. The feasible region is a simplex defined by $x_1 + x_2 + x_3 = 10$ and can be represented as follows. (x_1, x_2, and x_3 are the flows on the three paths.)

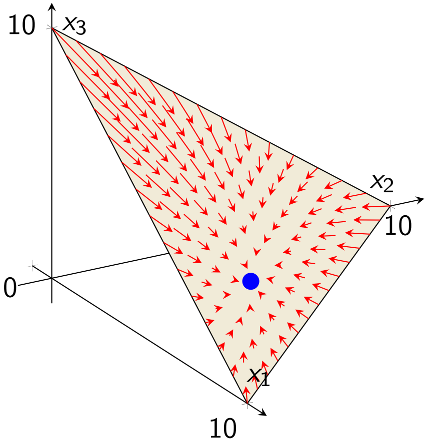


As before, the force field (payoffs) at any point is the vector $-\mathbf{t}(\mathbf{x})$. This can be projected on the simplex to get the projected payoffs.

Visualizing Equilibria

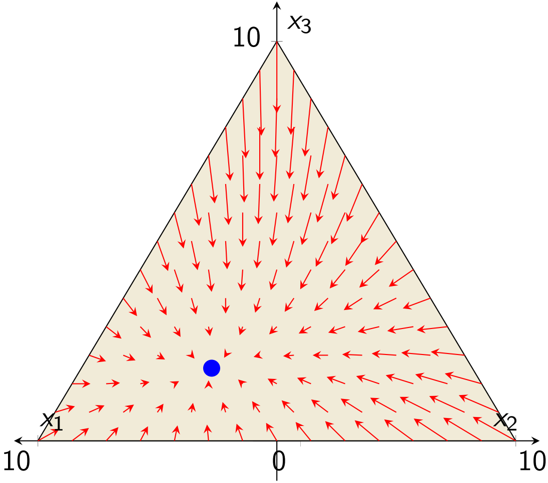
Three-path Network

The projected payoffs can be mapped on a 2D simplex. Equilibrium states are points at which an object remains stationary under the projected force field.



Visualizing Equilibria

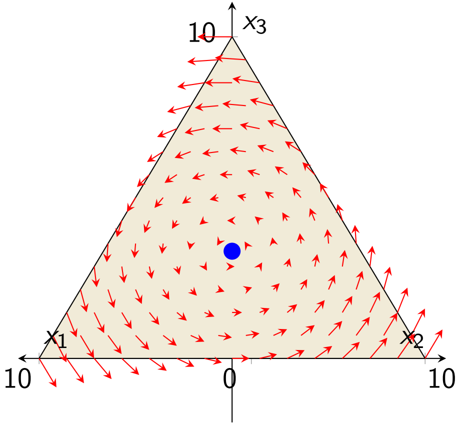
Three-path Network



Visualizing Equilibria

Three-path Network

Here's another example of the projected payoffs in which the equilibrium occurs at (10/3,10/3,10/3).



Visualizing Equilibria
Food for Thought

Some follow-up questions:

1. Does an equilibrium always exist?
2. If it does, is it unique or do multiple equilibria exist? Can we compute the equilibrium?
3. Also, we **guessed** that using negative of the path travel times, we can study equilibria just like we did in static mechanics. Are these equilibrium points Wardrop equilibria?

We will address the first and third questions in this lecture using fixed points and variational inequalities respectively.

Solutions to the second question will be discussed in later lectures.

Lecture Outline

Fixed Points

Fixed Points

Intuition

In the Braess network, we discussed how an equilibrium might evolve over multiple days from route switching behavior.

Suppose this flow shifting process was captured by a function $f : X \to X$, where X is the **set of path flows**. Imagine that f gives us the flow on the next day if we give it the current day's flow.

An equilibrium can be interpreted as a point at which the function returns the same flow pattern. Such a solution is what is called a **fixed point**.

Fixed Points

Definition

Definition (Fixed Point)

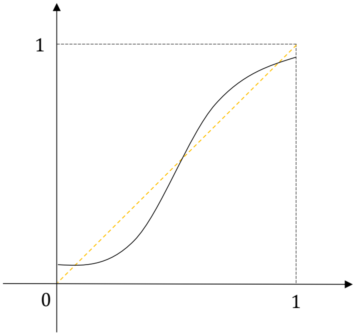
Let $f : X \to X$ be a function. A fixed point of f is a value $x \in X$ that satisfies $f(x) = x$.

One can determine conditions on X and f under which fixed points exist. Such results are called fixed point theorems.

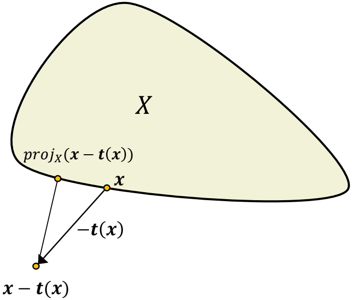
Theorem (Brouwer's Fixed Point Theorem)

Suppose $f : X \to X$ is a continuous function and let $X \subseteq \mathbb{R}^n$ be a compact convex set. Then f has at least one fixed point.

A set that is closed and bounded is compact.



To apply Brouwer's theorem, let us define our flow-shifting function f as $f(\mathbf{x}) = \text{proj}_X(\mathbf{x} - \mathbf{t}(\mathbf{x}))$, where the projection function of a point gives the nearest point in X (we'll formally define this later).

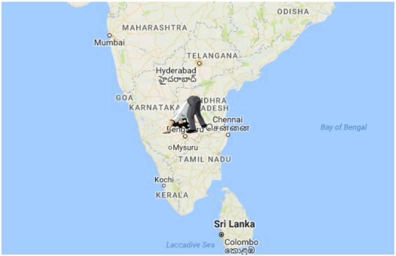


If $\mathbf{t}(\mathbf{x})$ is continuous, the projection function is continuous and the conditions of Brouwer's theorem are satisfied! In such cases, equilibria exist.

Remember that the theorem only tells us that under some assumptions on the travel time mappings, at least one equilibrium exists.

Definition (Antipodes)

The antipode of a point on the surface of the earth is a point that is diametrically opposite to it.

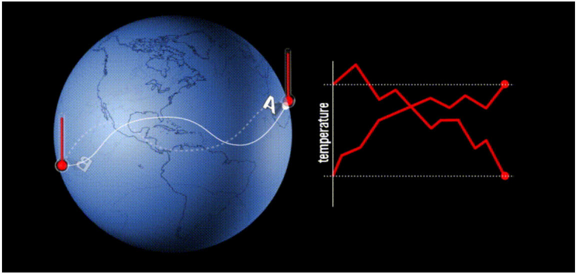


Source: https://www.antipodesmap.com/

Fixed Points

Trivia Break

There exists at least one pair of antipodes on the earth with the same temperature! (Why?)



Source: https://www.brilliant.org/

The above observation can be generalized to a fixed point-like theorem called the Borsuk-Ulam Theorem. Also check out https://www.youtube.com/watch?v=FhSFkLhDANA

Variational Inequalities

Variational Inequalities

Introduction

We hypothesized that a force field $-\mathbf{t}(\mathbf{x})$ will help us identify the equilibria and we established the conditions needed for existence using fixed point theory.

But why does it work? Can we formally prove that the equilibrium points satisfy Wardrop's principle?

We will address this problem in two steps. First, we will show that Fixed points $\equiv$ Variational Inequalities and then prove that Variational Inequalities $\equiv$ Wardrop Equilibria.

Definition (Variational Inequality)

Let $X \subseteq \mathbb{R}^n$ be a closed convex set and $\mathbf{f} : X \to \mathbb{R}^n$ be a continuous mapping. The finite-dimensional variational inequality problem $\mathrm{VI}(\mathbf{f}, X)$ is to find a vector $\mathbf{x}^*$ such that

$$\mathbf{f}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \,\forall\, \mathbf{x} \in X$$

Note that the definition is equivalent to $-\mathbf{f}(\mathbf{x}^*) \in \mathcal{N}_X(\mathbf{x}^*)$. But VIs are more general than the necessary and sufficient conditions for convex programs. (Why?)

VIs were first used in the 60s by Italian mathematician Guido Stampacchia to study partial differential equations for problems arising in mechanics.

Variational Inequalities

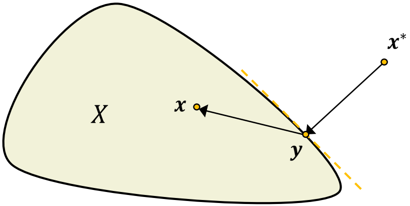
Connections with Fixed Points

Definition (Projection)

Let $X \subseteq \mathbb{R}^n$ be a closed convex set. For each $\mathbf{x}^* \in \mathbb{R}^n \exists ! \mathbf{y} \in X$ such that

$$\mathbf{y} = \arg\min_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{x}^*\|$$

$\mathbf{y}$ is called the projection of $\mathbf{x}^*$ on X and is denoted by $\text{proj}_X(\mathbf{x}^*)$.



Lemma

Let $X \subseteq \mathbb{R}^n$ be a closed convex set

$$\mathbf{y} = proj_X(\mathbf{x}^*) \Leftrightarrow (\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \geq 0 \, \forall \, \mathbf{x} \in X$$

Proof.

By definition, $\mathbf{y}$ minimizes $\|\mathbf{x} - \mathbf{x}^*\|$. Hence, it also minimizes $\|\mathbf{x} - \mathbf{x}^*\|^2$.

$\|\mathbf{x} - \mathbf{x}^*\|^2$ is convex in $\mathbf{x}$ and hence the necessary and sufficient conditions for optimality are

$$-2(\mathbf{y} - \mathbf{x}^*) \in \mathcal{N}_X(\mathbf{y})$$
$$-2(\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \leq 0 \, \forall \, \mathbf{x} \in X$$

∎

Proposition

Suppose X is closed and convex. $\mathbf{x}^*$ is a solution to $VI(\mathbf{f}, X)$ iff $\mathbf{x}^*$ is a fixed point of $proj_X(\mathbf{x} - \mathbf{f}(\mathbf{x}))$, i.e., $\mathbf{x}^* = proj_X(\mathbf{x}^* - \mathbf{f}(\mathbf{x}^*))$

Proof.

($\Rightarrow$) Since $\mathbf{x}^*$ is a solution to $\mathrm{VI}(\mathbf{f}, X)$,

$$f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \,\forall\, \mathbf{x} \in X$$
$$\Rightarrow \left(\mathbf{x}^* - (\mathbf{x}^* - f(\mathbf{x}^*))\right)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \,\forall\, \mathbf{x} \in X$$

According to previous lemma,

$$\mathbf{y} = \mathrm{proj}_X(\mathbf{x}^*) \Leftrightarrow (\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \geq 0 \,\forall\, \mathbf{x} \in X$$

Hence, $\mathbf{x}^* = \mathrm{proj}_X(\mathbf{x}^* - \mathbf{f}(\mathbf{x}^*))$.

($\Leftarrow$) Exercise. ∎

So far, we have established that

1. If $\mathbf{t}(\mathbf{x})$ is continuous, the function $\text{proj}_X(\mathbf{x} - \mathbf{t}(\mathbf{x}))$ has fixed points.
2. These fixed points solve $\text{VI}(\mathbf{t}, X)$.

The last piece of the puzzle is to prove that the solutions to the VI are actually Wardrop equilibria.

Theorem

$\mathbf{x}^*$ *satisfies the* $VI(\mathbf{t}, X) \Leftrightarrow$ *it satisfies the Wardrop principle*

Proof.

($\Rightarrow$) Since $\mathbf{x}^*$ satisfies the VI, $\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0$, i.e,

$$\mathbf{t}(\mathbf{x}^*)^T\mathbf{x}^* \leq \mathbf{t}(\mathbf{x}^*)^T\mathbf{x}\,\forall\,\mathbf{x} \in X$$

Imagine the path travel times are fixed at $\mathbf{t}(\mathbf{x}^*)$. The RHS, $\mathbf{t}(\mathbf{x}^*)^T\mathbf{x}$ is the total system travel time (TSTT) incurred by the flow pattern $\mathbf{x}$.

When the path travel times are fixed, TSTT is minimized if we route travelers on least travel time paths between each OD pair. Thus, from the above inequality $\mathbf{x}^*$ is a Wardrop equilibrium.

($\Leftarrow$) Exercise. ∎

Historical Notes

Supplementary Reading

The VI version for Wardrop equilibria was first discovered by Michael Smith in his 1979 seminal paper. [PDF]

Connections with the theory of VIs was formally identified an year later by Stella Dafermos* who also extended the conditions under which equilibria exist and provided an algorithm to compute it. [PDF]

Related text books:

- Patricksson, Chapter 3
- Nagurney, A. (2013). Network economics: A variational inequality approach (Vol. 10). Springer Science & Business Media.
- Sandholm, W. H. (2010). Population games and evolutionary dynamics. MIT press.

*She was the second woman to receive a PhD in Operations Research

Your Moment of Zen