CE 272
Traffic Network Equilibrium

Lecture 25

Sensitivity Analysis - Part I

Previously on Traffic Network Equilibrium...

Models in which a subset of links are tolled are called second-best pricing schemes. Since it is a constrained version we are not guaranteed to achieve an SO.

These models can be viewed as Stackelberg games in which the leader is the toll operator and the travelers are the followers.

Optimal tolls in such settings are harder to compute and the following methods are typically used.

- Bilevel Programs
- Mathematical Programs with Equilibrium Constraints (MPECs)

Previously on Traffic Network Equilibrium...

Bilevel programs have two optimization models: one at the upper level and another at the lower level.

For instance, at the upper level, the objective could be

$$\min_{\mathbf{c}} \sum_{(i,j)\in A} x_{ij}(\mathbf{c}) t_{ij}(x_{ij}(\mathbf{c}))$$

$$\text{s.t. } c_{ij} = 0 \, \forall \, (i,j) \in A'$$

where A' is the set of links on which tolls cannot be collected and $x_{ij}(\mathbf{c})$s are the solution to the lower level UE problem

$$\min_{\mathbf{x}\in X} \sum_{(i,j)\in A} \int_0^{x_{ij}} \left(t_{ij}(\omega) + c_{ij}\right) \, d\omega$$

Note that the c values are assumed to be fixed in the lower level.

Previously on Traffic Network Equilibrium...

Bush-based methods essentially solve the following version of the TAP. The decision variables are x_{ij}^r, the link flows segregated by origins.

$$\min \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in A} x_{ij}^r - \sum_{h:(h,i)\in A} x_{hi}^r = \begin{cases} \sum_{s\in Z} d_{is} & \text{if } i = r \\ -d_{ri} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \quad \forall\, r \in Z$$

$$x_{ij} = \sum_{r\in Z} x_{ij}^r \,\forall\, (i,j) \in A$$

$$x_{ij}^r \geq 0 \,\forall\, (i,j) \in A, r \in Z$$

We will discuss two main components of the algorithm today and put these pieces together in the next class.

Lecture Outline

1. Sensitivity to Demand
2. OD Matrix Estimation

Lecture Outline

Sensitivity to Demand

Sensitivity to Demand

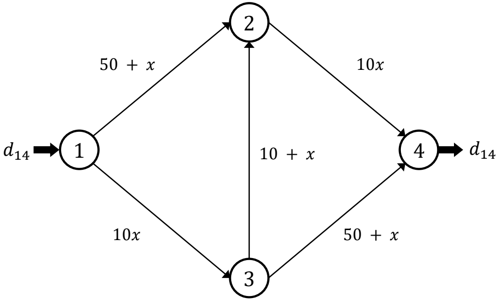
Introduction

Sensitivity analysis broadly refers to the change in model outputs with change in inputs.

In today's class and the next, we will look at how the equilibrium solutions vary as the demand and link parameters change.

We will also look at some applications such as OD matrix estimation and network design to understand the importance of sensitivity analysis.

In the first assignment, you solved for the UE solutions for different demands in the Braess network.

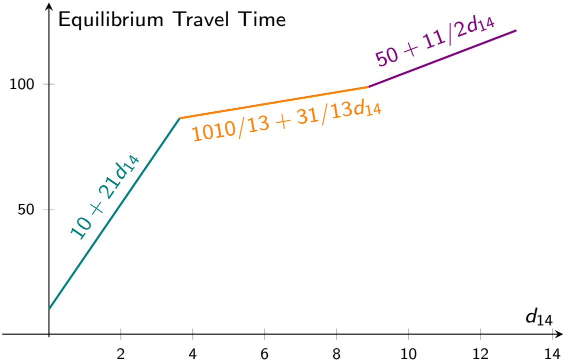


Can you plot a graph with demand on the x-axis and UE travel time on the y-axis? What happens at very low/med/high demand?

Sensitivity to Demand

Introduction

The plot has three regimes, each corresponding to different sets of used paths.



- The plot is a piecewise linear function. Is it always linear?
- The equilibrium travel times are differentiable within each of these pieces.
- At the degenerate points, the function is not differentiable and unused paths have same travel times as that of the used ones.

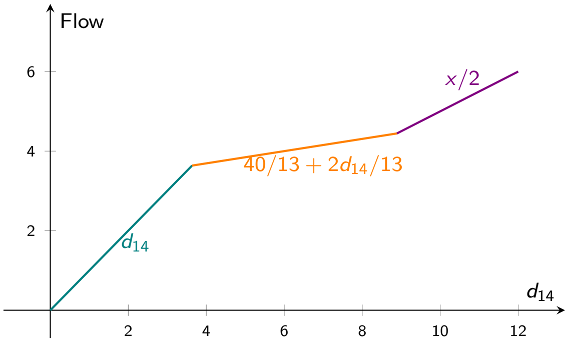
Sensitivity to Demand

Introduction

Construct plots for equilibrium link flows for different values of d_{14}.

Sensitivity to Link Parameters

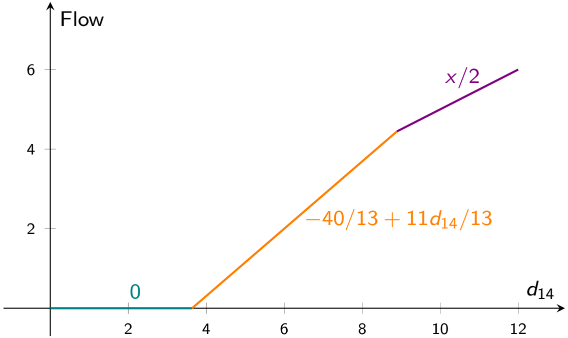
Introduction



Flow on (1,3) and (2,4)

Sensitivity to Link Parameters

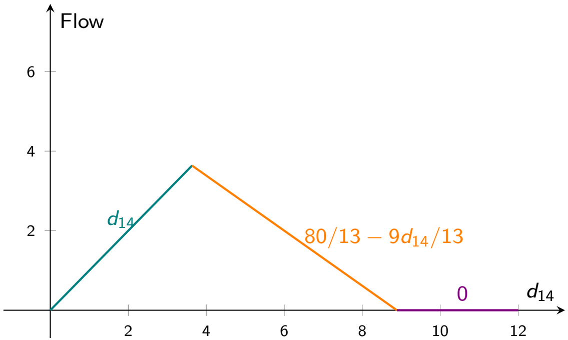
Introduction



Flow on (1,2) and (3,4)

Sensitivity to Link Parameters

Introduction



Flow on (3,2)

Sensitivity to Demand

Introduction

Sensitivity analysis involves finding the derivatives of these functions (i.e., equilibrium travel times and link flows) with respect to input parameters at a given solution.

To find these sensitivities, we assume that the current solution is not degenerate and hence derivatives exist.

In other words, we assume that, at the given solution, all shortest paths have strictly positive flows.

Sensitivity to Demand

Bush-based Approach

Sensitivity of equilibrium solutions can be handled using matrix, variational inequalities, or bush-based methods.

In the bush-based approach, since the current solution is non-degenerate, if $x_{ij}^r = 0$, then the link (i, j) does not belong to any shortest path from r.

Stated differently, $x_{ij}^r > 0 \Leftrightarrow \mu_j^r = \mu_i^r + t_{ij}(x_{ij})$, where x_{ij}^r is the current equilibrium flow solution and μ's represent the shortest path labels.

Furthermore, since the solution is non-degenerate, it is assumed that minor changes in demand will not change the equilibrium bushes (set of used paths remain same).

We could use a superscript * to denote that these are equilibrium flows, but we'll avoid that for brevity.

Sensitivity to Demand

Bush-based Approach

The equilibrium solution to the bush based-formulation must satisfy

$$\mu_r^r = 0 \, \forall \, r \in Z$$

$$\mu_j^r - \mu_i^r - t_{ij}(x_{ij}) = 0 \, \forall \, (i,j) \in B^r, r \in Z$$

$$\sum_{j:(i,j)\in A} x_{ij}^r - \sum_{h:(h,i)\in A} x_{hi}^r = \begin{cases} \sum_{s\in Z} d_{is} & \text{if } i = r \\ -d_{ri} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \quad \forall \, r \in Z$$

$$x_{ij}^r = 0 \, \forall \, (i,j) \notin B^r, r \in Z$$

When the demand is perturbed, the bushes remain same. Hence, these conditions must hold for the new equilibrium values of x and μ.

Sensitivity to Demand

Bush-based Approach

Suppose the demand between an OD pair $(u, v) \in Z^2$ changes. Let

$$\xi_{ij}^r = \frac{\partial x_{ij}^r}{\partial d_{uv}}, \Lambda_i^r = \frac{\partial \mu_i^r}{\partial d_{uv}}$$

Differentiate the following equations with respect to d_{uv}.

$$\mu_r^r = 0 \,\forall\, r \in Z$$

$$\mu_j^r - \mu_i^r - t_{ij}(x_{ij}) = 0 \,\forall\, (i,j) \in B^r, r \in Z$$

$$\sum_{j:(i,j)\in A} x_{ij}^r - \sum_{h:(h,i)\in A} x_{hi}^r = \begin{cases} \sum_{s\in Z} d_{is} & \text{if } i = r \\ -d_{ri} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \quad \forall\, r \in Z$$

$$x_{ij}^r = 0 \,\forall\, (i,j) \notin B^r, r \in Z$$

Sensitivity to Demand

Bush-based Approach

Suppose the demand between an OD pair $(u, v) \in Z^2$ changes. Let

$$\xi_{ij}^r = \frac{\partial x_{ij}^r}{\partial d_{uv}}, \Lambda_i^r = \frac{\partial \mu_i^r}{\partial d_{uv}}$$

The derivatives must satisfy

$$\Lambda_r^r = 0 \,\forall\, r \in Z$$

$$\Lambda_j^r - \Lambda_i^r - t'_{ij}(x_{ij}) \sum_{r' \in Z} \xi_{ij}^{r'} = 0 \,\forall\, (i,j) \in B^r, r \in Z$$

$$\sum_{j:(i,j)\in A} \xi_{ij}^r - \sum_{h:(h,i)\in A} \xi_{hi}^r = \begin{cases} 1 & \text{if } i = u, r = u \\ -1 & \text{if } i = v, r = u \;\; \forall\, r \in Z \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_{ij}^r = 0 \,\forall\, (i,j) \notin B^r, r \in Z$$

This is a linear system of equations!

Sensitivity to Demand

Bush-based Approach

Alternately, notice that these form the KKT conditions of the following optimization model.

$$\min \sum_{(i,j)\in A} \int_0^{\xi_{ij}} t'_{ij}(x_{ij})\xi d\xi$$

$$\sum_{j:(i,j)\in A} \xi^r_{ij} - \sum_{h:(h,i)\in A} \xi^r_{hi} = \begin{cases} 1 & \text{if } i = u, r = u \\ -1 & \text{if } i = v, r = u \ \forall\, r \in Z \\ 0 & \text{otherwise} \end{cases}$$

$$\xi^r_{ij} = 0 \,\forall\, (i,j) \notin B^r, r \in Z$$

The optimal values of Λs can be obtained by solving a shortest path problem with fixed link travel times $t'_{ij}(x_{ij})\xi^*_{ij}$, where ξ^* is a solution to the above formulation.

Sensitivity to Demand

Bush-based Approach

This problem is essentially an 'equilibrium formulation' with linear link delay functions and 1 unit of demand between u and v.

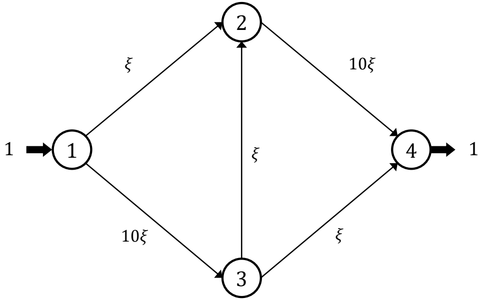
$$\min \sum_{(i,j)\in A} \int_0^{\xi_{ij}} t'_{ij}(x_{ij})\xi d\xi$$

$$\sum_{j:(i,j)\in A} \xi^r_{ij} - \sum_{h:(h,i)\in A} \xi^r_{hi} = \begin{cases} 1 & \text{if } i = u, r = u \\ -1 & \text{if } i = v, r = u \;\forall\, r \in Z \\ 0 & \text{otherwise} \end{cases}$$

$$\xi^r_{ij} = 0 \,\forall\, (i,j) \notin B^r, r \in Z$$

However, there are no non-negativity constraints! The derivatives can take negative values (as observed in the Braess network).

Thus, to compute the sensitivity to demand between $(1, 4)$, solve the UE problem in the following network.

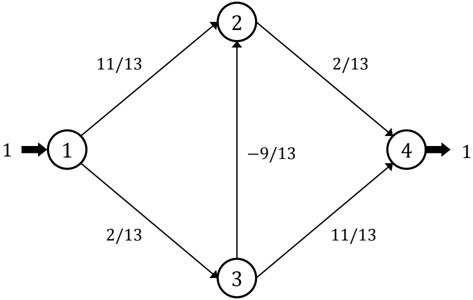


Recall that all links are used in the original UE solution. If not, we should only use the equilibrium bush links to solve this new ‘UE’ problem.

Sensitivity to Demand

Bush-based Approach

The optimal ξ values are shown below. What is the 'equilibrium travel time' in the following network?



The 'UE travel time' in the above figure corresponds to the slope of the piecewise linear function between the UE travel time and demand.

OD Matrix Estimation

OD Matrix Estimation

Introduction

Let us now look at an application of the sensitivity analysis method discussed so far.

Typically, the OD matrices or trip tables that you have been using are developed from a 4-step process or activity-based methods.

These methods are very involved and may be error prone or outdated. In such instances, one could use data from the field to improve our estimates of the OD matrix.

OD Matrix Estimation

Introduction

The easiest traffic data that can be collected are link-level volumes. Can we estimate the OD flows if we have access to link volumes?

What if we have access to flows on all links? The OD solutions are still indeterminate.

Typically, it is assumed that an incorrect estimate of OD $\hat{\mathbf{d}}$ is known.

The objective is to then optimize the following problem.

$$\min f(\mathbf{d},\mathbf{x}) = \gamma \sum_{(r,s)\in Z^2} (d_{rs} - \hat{d}_{rs})^2 + (1-\gamma) \sum_{(i,j)\in \hat{A}} (x_{ij} - \hat{x}_{ij})^2$$

$$\text{s.t. } \mathbf{x} \in \arg \min_{\mathbf{x}\in X(\mathbf{d})} \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x)dx$$

$$d_{rs} \geq 0 \, (r,s) \in Z^2$$

- γ is a parameter that reflects the confidence we have in our link flows or initial OD estimate.
- For a given **d**, the constraint provides an equilibrium link flow solution **x**. Hence, we can write this as an optimization problem purely in terms of **d**, i.e., $f(\mathbf{d},\mathbf{x}(\mathbf{d}))$.

OD Matrix Estimation

Formulation

Without the equilibrium constraints, the model has only non-negativity constraints.

Hence, we could use a gradient projection-like method to adjust the **d** values and project it to the positive orthant.

$$\mathbf{d} \leftarrow [\mathbf{d} - \eta \nabla f]^+$$

To this end, we need the derivatives of the objective with respect to **d**. This is where sensitivity analysis comes handy.

$$\min f(\mathbf{d}, \mathbf{x}) = \eta \sum_{(r,s) \in Z^2} (d_{rs} - \hat{d}_{rs})^2 + (1 - \eta) \sum_{(i,j) \in \hat{A}} (x_{ij} - \hat{x}_{ij})^2$$

The derivative of this function with respect to d_{rs} is given by

$$\frac{\partial f}{\partial d_{rs}} = 2\eta(d_{rs} - \hat{d}_{rs}) + 2(1 - \eta) \sum_{(i,j) \in \hat{A}} (x_{ij} - \hat{x}_{ij}) \frac{\partial x_{ij}}{\partial d_{rs}}$$

The values of $\frac{\partial x_{ij}}{\partial d_{rs}}$ can be obtained from the ξ values of the sensitivity analysis method.

Note that we need to solve a total of $|Z^2|$ equilibrium problems to obtain the gradient of the above function.

Your Moment of Zen

Science Smiles

R K Laxman

I am planning for the future!
There will be traffic jam in space!