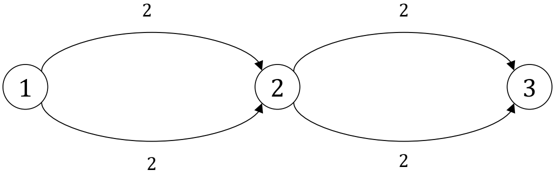
CE 272
Traffic Network Equilibrium

Lecture 21

Entropy Maximization

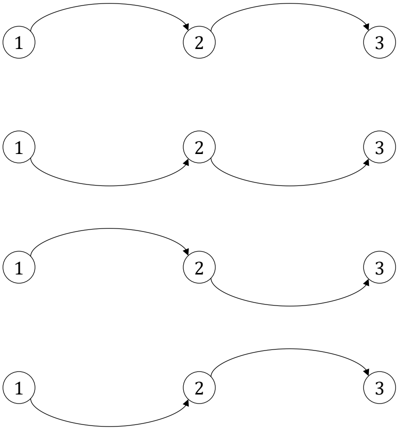
Previously on Traffic Network Equilibrium...

Given a link flow vector **x**, the path flows cannot however be uniquely identified.



Can you find multiple path flows in the above network?

Previously on Traffic Network Equilibrium...



Path Flow 1	Path Flow 2	Path Flow 3
1	2	0
1	2	0
1	0	2
1	0	2

Lecture Outline

Lecture Outline

Introduction

Introduction

Applications of Path Flows

If the traffic assignment problem is used only for predicting congestion, then the link flows and link travel times suffice.

However, many interesting applications benefit from the knowledge of predicted routes of all travelers.

Introduction

Applications of Path Flows

In Lecture 1, we saw one such example: Select link analysis, which helps identify the OD pairs that use a particular link.

Other applications of path flow solutions include

- Estimating emissions
- Predicting OD flows from link-level counts
- Path differentiated congestion pricing
- Equity analysis

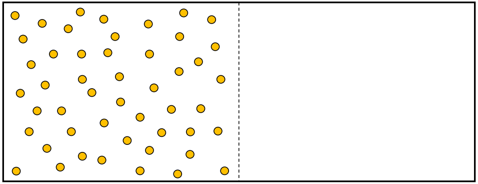
Introduction

Intuition

Since multiple path flow solutions exist for a given link flow pattern, we will identify the solution that we think is *most likely*.

Before proceeding to identify the most likely path flow, let us study a related example.

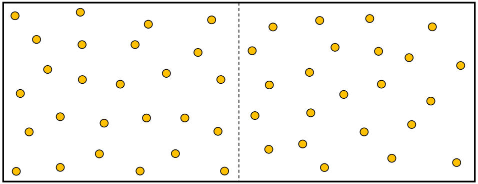
Imagine a container with n gas molecules. Suppose, at some time instance, they are all present at one end of the container.



What is the probability with which this configuration can occur?

The probability with which each molecule could be in the left half is $1/2$. Hence, the probability of above configuration is $(1/2)^n$.

Now consider an alternate configuration. Do you think this is more likely?



What is the probability of this new configuration?

Suppose we toss a coin and if we see heads, we place the molecule in the left portion and we see tails, we place it at the right end.

For the new configuration, we need exactly $n/2$ heads in n trials, where the probability of heads is $1/2$. Using the pmf of the Binomial distribution, the odds of the configuration is

$$\binom{n}{n/2}\left(\frac{1}{2}\right)^{n/2}\left(\frac{1}{2}\right)^{n/2} = \frac{n!}{\left((n/2)!\right)^2}\left(\frac{1}{2}\right)^{n}$$

As $n! \ggg \left((n/2)!\right)^2$, the above probability is greater than the probability of the previous configuration.

The second configuration is said to have higher *entropy* compared to the first. In other words, it is a more disordered state.

Likewise, we will try to find a high entropy path flow solution that spreads out travelers across minimal paths.

Does the GP algorithm provide high entropy solutions?

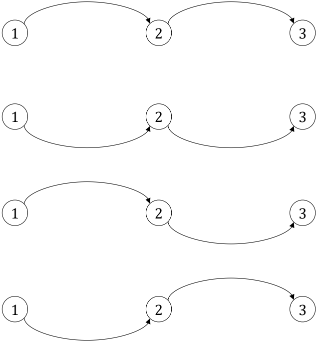
Lecture Outline

Entropy Maximization

Entropy Maximization

Introduction

Which of these three solutions has the highest entropy?



Path Flow 1	Path Flow 2	Path Flow 3
1	2	0
1	2	0
1	0	2
1	0	2

Imagine k paths between a single OD pair. Suppose, there are $y_1, y_2, \ldots, y_k$ travelers on these paths and for now assume they are integers.

Roll a k-sided die and assign all d travelers to different paths. The probability that exactly $y_1, y_2, \ldots, y_k$ travelers are on these paths is given by the pmf of the multinomial distribution

$$\frac{d!}{y_1!y_2!\ldots y_k!}\left(\frac{1}{k}\right)^{y_1}\left(\frac{1}{k}\right)^{y_2}\ldots\left(\frac{1}{k}\right)^{y_k} = \frac{d!}{y_1!y_2!\ldots y_k!}\left(\frac{1}{k}\right)^{d}$$

Hence, a path flow solution has the largest entropy if $\frac{d!}{y_1!y_2!\ldots y_k!}$ is maximized.

Instead of maximizing the factorial terms, we can maximize the logarithm of the above expression

$$\ln d! - \sum_{p=1}^{k} \ln y_p!$$

which can be approximated using Stirling's formula as

$$\begin{aligned}
&(d \ln d - d) - \sum_{p=1}^{k} (y_p \ln y_p - y_p) \\
&= \sum_{p=1}^{k} y_p \ln d - d - \sum_{p=1}^{k} y_p \ln y_p + \sum_{p=1}^{k} y_p \\
&= \sum_{p=1}^{k} y_p (\ln d - \ln y_p) = -\sum_{p=1}^{k} y_p \ln(y_p / d)
\end{aligned}$$

Entropy Maximization

Extension to Multiple OD Pairs

For multiple OD pairs, the probability of observing a given flow pattern is the product of probabilities of observing the flows for individual OD pairs.

Thus, Stirling's approximation will result in an extra summation across all OD pairs:

$$-\sum_{(r,s)\in Z^2}\sum_{p\in P_{rs}} y_p \ln(y_p/d_{rs})$$

If we include constraints in which the path flow variables provide equilibrium link flows (which are unique), then we can write the entropy maximization problem as a non-linear program.

Entropy Maximization

Optimization Formulation

The x_{ij}^{UE}s in the formulation are the UE link flows (from MSA or FW) and are constants.

$$
\begin{aligned}
\min \ & \sum_{(r,s)\in Z^2} \sum_{p \in P_{rs}} y_p \ln(y_p/d_{rs}) \\
\text{s.t. } & \sum_{p \in P} \delta_{ij}^p y_p = x_{ij}^{UE} \ \forall \ (i,j) \in A \\
& \sum_{p \in P_{rs}} y_p = d_{rs} \ \forall \ (r,s) \in Z^2 \\
& y_p \geq 0 \ \forall \ p \in P
\end{aligned}
$$

Is the objective convex? What's the challenge in solving the above convex program?

Proportionality

Proportionality

Introduction

It so happens that maximizing entropy guarantees that the path flows satisfy a property called proportionality (which we will derive using the KKT conditions).

This property is exploited in some of the most recent equilibrium algorithms.

Proportionality

KKT Conditions

Write the KKT conditions of the following program.

$$
\begin{aligned}
\min \quad & \sum_{(r,s)\in Z^2} \sum_{p\in P_{rs}} y_p \ln(y_p/d_{rs}) \\
\text{s.t.} \quad & \sum_{p\in P} \delta_{ij}^p y_p = x_{ij}^{UE} \ \forall \ (i,j) \in A \\
& \sum_{p\in P_{rs}} y_p = d_{rs} \ \forall \ (r,s) \in Z^2 \\
& y_p \geq 0 \ \forall \ p \in P
\end{aligned}
$$

Note that the non-negativity constraints in the maximum entropy formulation are redundant.

Proportionality

KKT Conditions

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{(r,s)\in Z^2} \sum_{p\in P_{rs}} y_p \ln(y_p/d_{rs}) + \sum_{(i,j)\in A} \lambda_{ij} \left(x_{ij}^{UE} - \sum_{p\in P} \delta_{ij}^p y_p \right) + \sum_{(r,s)\in Z^2} \mu_{rs} \left(d_{rs} - \sum_{p\in P_{rs}} y_p \right)$$

Proportionality

KKT Conditions

Primal feasibility:

$$\sum_{p \in P} \delta_{ij}^{p} y_p = x_{ij}^{UE} \ \forall \ (i,j) \in A$$

$$\sum_{p \in P_{rs}} y_p = d_{rs} \ \forall \ (r,s) \in Z^2$$

Dual feasibility:

Complementary Slackness:

Gradient of the Lagrangian vanishes:

$$y_p \frac{\partial}{\partial y_p} \ln(y_p / d_{rs}) + \ln(y_p / d_{rs}) - \sum_{(i,j) \in A} \delta_{ij}^{p} \lambda_{ij} - \mu_{rs} = 0 \ \forall \ p \in P_{rs}, (r,s) \in Z^2$$

Proportionality

KKT Conditions

The last condition can be written as

$$1 + \ln(y_p/d_{rs}) - \sum_{(i,j)\in A} \delta_{ij}^{p}\lambda_{ij} - \mu_{rs} = 0$$

$$\Rightarrow y_p = d_{rs} \exp\Big(-1 + \mu_{rs} + \sum_{(i,j)\in A} \delta_{ij}^{p}\lambda_{ij}\Big)$$

From supply-demand constraint/primal feasibility, for an OD pair (r, s),

$$\sum_{p\in P_{rs}} d_{rs} \exp\Big(-1 + \mu_{rs} + \sum_{(i,j)\in A} \delta_{ij}^{p}\lambda_{ij}\Big) = d_{rs}$$

$$\sum_{p\in P_{rs}} \exp(-1 + \mu_{rs}) \exp\Big(\sum_{(i,j)\in A} \delta_{ij}^{p}\lambda_{ij}\Big) = 1$$

$$\sum_{p\in P_{rs}} \exp\Big(\sum_{(i,j)\in A} \delta_{ij}^{p}\lambda_{ij}\Big) = \exp(1 - \mu_{rs})$$

Proportionality
KKT Conditions

We can thus write μ_{rs} as a function of λs as follows:

$$\mu_{rs} = 1 - \ln\left(\sum_{p \in P_{rs}} \exp\left(\sum_{(i,j)\in A} \delta_{ij}^p \lambda_{ij}\right)\right)$$

Plugging this in the last KKT condition,

$$\begin{aligned} y_p &= d_{rs} \exp\left(\sum_{(i,j)\in A} \delta_{ij}^p \lambda_{ij} - \ln\left(\sum_{p' \in P_{rs}} \exp\left(\sum_{(i,j)\in A} \delta_{ij}^{p'} \lambda_{ij}\right)\right)\right) \\ &= \frac{d_{rs}}{\sum_{p' \in P_{rs}} \exp\left(\sum_{(i,j)\in A} \delta_{ij}^{p'} \lambda_{ij}\right)} \exp\left(\sum_{(i,j)\in A} \delta_{ij}^p \lambda_{ij}\right) \end{aligned}$$

The first term in the above expression depends only on the OD pair (r, s). (Why?)

Proportionality

KKT Conditions

Hence, we may write

$$y_p = K_{rs} \exp\Big(\sum_{(i,j)\in A} \delta_{ij}^p \lambda_{ij}\Big)$$

where K_{rs} is some constant. Further, for any paths p and p' between (r,s),

$$\begin{aligned}\frac{y_p}{y_{p'}} &= \frac{\exp\Big(\sum_{(i,j)\in A} \delta_{ij}^p \lambda_{ij}\Big)}{\exp\Big(\sum_{(i,j)\in A} \delta_{ij}^{p'} \lambda_{ij}\Big)} \\ &= \frac{\prod_{(i,j)\in \hat{A}} \exp(\delta_{ij}^p \lambda_{ij})}{\prod_{(i,j)\in \bar{A}} \exp(\delta_{ij}^{p'} \lambda_{ij})}\end{aligned}$$

where $\hat{A}$ is the set of links that belong to p and not p' and $\bar{A}$ is the set of links that belong to p' and not p. The above ratio depends only on the pairs of alternate segments (PAS) and not on the OD pair!

Proportionality

KKT Conditions

The converse of the above observation is not true! In other words, flow solutions that satisfy proportionality need not maximize entropy.

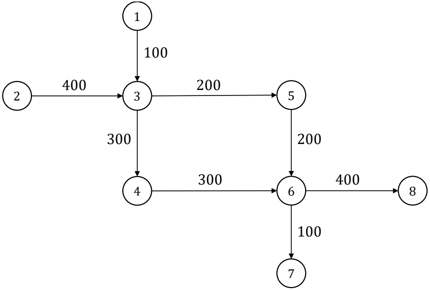
Network	# MEUE optimality conditions = # basic DSPRs			
	EWO[a]	EBO[b]	Core	Total (A)
Sioux Falls	138	70	5	213
Barcelona	2,268	948	38	3,254
Winnipeg	2,430	2,942	2	5,374
Tucson	1,179,105	22,911	8	1,202,024
Chicago s.	27,261	6,204	4	33,469
Chicago r.	89,822,183	901,656	91	90,723,930

Proportionality drastically reduces the search space (see column Core in the above figure).

Proportionality

Example

Consider the following network. Suppose that the demand between 1 and 8 is 100 and between 2 and 7 is 200.



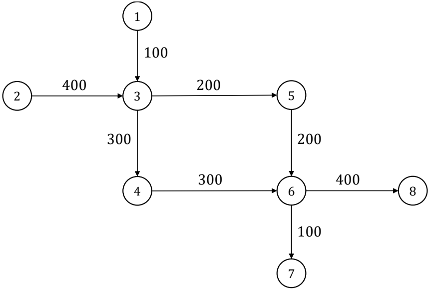
Travelers between each OD pair have two paths to choose from. The numbers on the links represent equilibrium flows.

What is an obvious path flow decomposition? Does it maximize entropy?

Proportionality

Example

The entropy maximizing solution splits travelers between the two paths in the following way.



- **OD Pair (1,7):**
  Path 1-3-5-6-7: 40
  Path 1-3-4-6-7: 60
- **OD Pair (2,8):**
  Path 2-3-5-6-8: 160
  Path 2-3-4-6-8: 240

Notice that the ratios of path flows on the PASs for both OD pairs are same! (Why?)

Proportionality

Over the last two lectures we saw that the equilibrium solutions satisfy two interesting properties.

1. The equilibrium OD flow cannot be present on both sides of a two-way street.
2. The ratio of flows on any two routes between an OD pair is independent of the OD demand. Further, for a given PAS, the ratio of flows on the two segments is same across all OD pairs.

Proportionality

Bush-based Algorithms and TAPAS

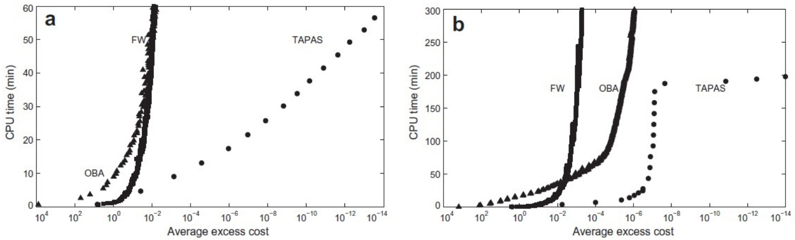
The first observation led to the development of bush-based algorithms which we will discuss over the next few lectures. The run-times of these are orders of magnitude faster than FW and GP.

The second observation led to the development of TAPAS (Traffic Assignment by Paired Alternate Segments) by Hillel Bar-Gera in 2010.

TAPAS is a bush-based method that finds equilibrium flows while ensuring proportionality. It is currently one of the fastest algorithms for finding the equilibrium solutions.

Proportionality

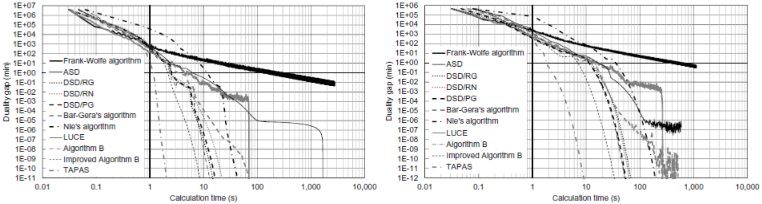
Bush-based Algorithms and TAPAS



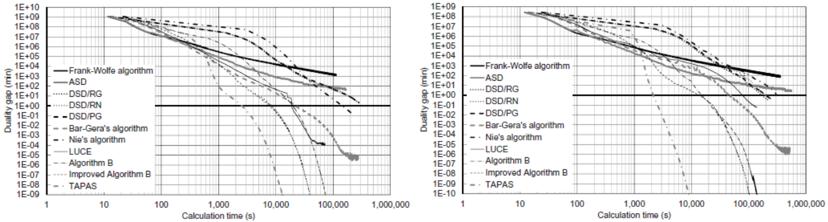
CPU time required to achieve desired levels of convergence. (a) Chicago regional; (b) Philadelphia.

Proportionality

Bush-based Algorithms and TAPAS



Computation results (left: Barcelona, right: Winnipeg)



Computation results (left: Philadelphia, right: Chicago regional network)

Suggested Reading

Bar-Gera, H. (2006). Primal method for determining the most likely route flows in large road networks. Transportation Science, 40(3), 269-286.

Bar-Gera, H. (2010). Traffic assignment by paired alternative segments. Transportation Research Part B: Methodological, 44(8-9), 1022-1046.

Inoue, S. I., & Maruyama, T. (2012). Computational experience on advanced algorithms for user equilibrium traffic assignment problem and its convergence error. Procedia-Social and Behavioral Sciences, 43, 445-456.

Your Moment of Zen