CE 272
Traffic Network Equilibrium

Lecture 14

Multi-Class User Equilibrium - Part II

Previously on Traffic Network Equilibrium...

The demand information for all OD pairs is commonly referred to as **OD matrix** or **trip tables**.

The number of person trips are computed from the first two steps of the four-step process. In the third step, these trips are assigned to different modes (car, bus, two-wheeler etc.) resulting in a trip table for each mode.

But for equilibrium analysis, we assume that demand comprises of only passenger cars. The demand of other types of vehicles are adjusted by factors called **passenger car units (PCUs)** that reflect their sizes relative to that of a car.

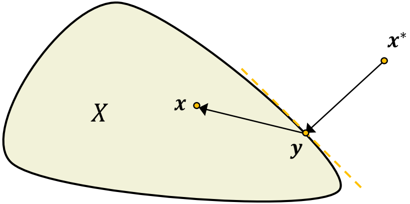
Previously on Traffic Network Equilibrium...

Definition (Projection)

Let $X \subseteq \mathbb{R}^n$ be a closed convex set. For each $\mathbf{x}^* \in \mathbb{R}^n \exists! \mathbf{y} \in X$ such that

$$\mathbf{y} = \arg\min_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{x}^*\|$$

$\mathbf{y}$ is called the projection of $\mathbf{x}^*$ on X and is denoted by $\text{proj}_X(\mathbf{x}^*)$.



Previously on Traffic Network Equilibrium...

Lemma

Let $X \subseteq \mathbb{R}^n$ be a closed convex set

$$\mathbf{y} = proj_X(\mathbf{x}^*) \Leftrightarrow (\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \geq 0 \, \forall \, \mathbf{x} \in X$$

Proof.

By definition, $\mathbf{y}$ minimizes $\|\mathbf{x} - \mathbf{x}^*\|$. Hence, it also minimizes $\|\mathbf{x} - \mathbf{x}^*\|^2$.

$\|\mathbf{x} - \mathbf{x}^*\|^2$ is convex in $\mathbf{x}$ and hence the necessary and sufficient conditions for optimality are

$$-2(\mathbf{y} - \mathbf{x}^*) \in \mathcal{N}_X(\mathbf{y})$$
$$-2(\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \leq 0 \, \forall \, \mathbf{x} \in X$$

■

Previously on Traffic Network Equilibrium...

Proposition

Suppose X is closed and convex. $\mathbf{x}^*$ is a solution to $VI(\mathbf{f}, X)$ iff $\mathbf{x}^*$ is a fixed point of $proj_X(\mathbf{x} - \mathbf{f}(\mathbf{x}))$, i.e., $\mathbf{x}^* = proj_X(\mathbf{x}^* - \mathbf{f}(\mathbf{x}^*))$

Proof.

$(\Rightarrow)$ Since $\mathbf{x}^*$ is a solution to $\mathrm{VI}(\mathbf{f}, X)$,

$$f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \, \forall \, \mathbf{x} \in X$$

$$\Rightarrow \left(\mathbf{x}^* - (\mathbf{x}^* - f(\mathbf{x}^*))\right)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \, \forall \, \mathbf{x} \in X$$

According to previous lemma,

$$\mathbf{y} = \mathrm{proj}_X(\mathbf{x}^*) \Leftrightarrow (\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \geq 0 \, \forall \, \mathbf{x} \in X$$

Hence, $\mathbf{x}^* = \mathrm{proj}_X(\mathbf{x}^* - \mathbf{f}(\mathbf{x}^*))$.

$(\Leftarrow)$ Exercise. ■

Previously on Traffic Network Equilibrium...

So far, we have established that

1. If $\mathbf{t}(\mathbf{x})$ is continuous, the function $\text{proj}_X(\mathbf{x} - \mathbf{t}(\mathbf{x}))$ has fixed points.
2. These fixed points solve $\text{VI}(\mathbf{t}, X)$.

The last piece of the puzzle is to prove that the solutions to the VI are actually Wardrop equilibria.

Theorem

$\mathbf{x}^*$ *satisfies the* $\mathit{VI}(\mathbf{t}, X) \Leftrightarrow$ *it satisfies the Wardrop principle*

Lecture Outline

1. Strict Monotonicty and Uniqueness
2. Optimization Model for Asymmetric TAP

Lecture Outline

Strict Monotonicty and Uniqueness

Strict Monotonicty and Uniqueness

For asymmetric TAPs, we will have to use VIs to model equilibrium flows. We've seen that VIs have a solution when $\mathbf{t}(\mathbf{x})$ is continuous, but we haven't discussed

- Uniqueness of VI solutions
- Algorithmic procedures to compute equilibria

We will first address these aspects by making an additional assumption called **strict monotonicity** and then study a more relaxed version of the problem.

Strict Monotonicty and Uniqueness

Definition and Properties

Definition

The travel time mapping $\mathbf{t}(\mathbf{x}) : X \subseteq \mathbb{R}^m \to \mathbb{R}^m$ is said to be *strictly monotone* if

$$\left(\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x}')\right)^T (\mathbf{x} - \mathbf{x}') > 0 \,\forall\, \mathbf{x}, \mathbf{x}' \in X, \mathbf{x} \neq \mathbf{x}'$$

Loosely speaking, strict monotonicity implies that the diagonal terms of the Jacobian are large compared to the off-diagonal terms.

Proposition

Let $\mathbf{t}(\mathbf{x})$ *be a continuous function on a convex domain.* $\mathbf{t}$ *is strictly monotone* $\Leftrightarrow$ *its Jacobian is positive definite.*

Proposition

If $\mathbf{t}$ *is strictly monotone and continuous, then a solution to the VI(*$\mathbf{t}, X$*) is unique.*

Strict Monotonicty and Uniqueness

Diagonalization

At each iteration, select an arc (i, j). Fix the flows of all the other arcs and construct a real-valued functions for the delay all the arcs in the network.

Suppose the flow at the kth iteration is $(x_1^k, x_2^k, \ldots, x_m^k)$. Solve the sub-problem in which the objective is

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x_1^k, x_2^k, \ldots, \omega, \ldots x_m^k)\, d\omega$$

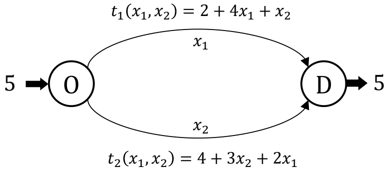
Let $\tilde{t}_{ij}(x_{ij}) = t_{ij}(x_1^k, x_2^k, \ldots, x_{ij}, \ldots x_m^k)$. Then, the step size η in the FW iteration is obtained by solving

$$\sum_{(i,j)\in A} \tilde{t}_{ij}(\eta \hat{x}_{ij} + (1-\eta)x_{ij})(\hat{x}_{ij} - x_{ij}) = 0$$

Strict Monotonicty and Uniqueness

Example

Consider the two-link network with link delay functions as shown below:



Suppose at the current iteration the flows are $(x_1, x_2) = (0, 5)$. The travel times are $(t_1, t_2) = (7, 19)$. Hence, the all-or-nothing solution is $(\hat{x}_1, \hat{x}_2) = (5, 0)$. The sub-problem is therefore to minimize

$$\int_0^{\eta\hat{x}_1+(1-\eta)x_1} (2 + 4\omega + 5)\, d\omega + \int_0^{\eta\hat{x}_2+(1-\eta)x_2} (4 + 3\omega)\, d\omega$$

$$= \int_0^{5\eta} (2 + 4\omega + 5)\, d\omega + \int_0^{5(1-\eta)} (4 + 3\omega)\, d\omega$$

Strict Monotonicty and Uniqueness

Convergence

Diagonalization method is known to converge to the equilibrium solution if the Jacobian of the travel time functions is positive definite.

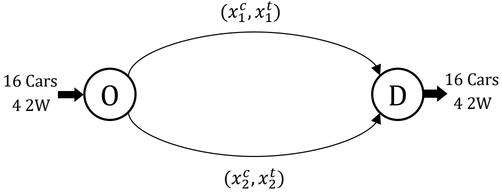
But for more general settings, it may not converge!

Strict Monotonicty and Uniqueness

General Asymmetric TAP

If strict monotonicity is not assumed, the traffic assignment problem can have multiple solutions.

Consider, the two link network with the two classes cars and two-wheelers.



Suppose that the link delay functions on link $i = 1, 2$ is given by

$$t_i^c(x_i^c, x_i^t) = 1.5x_i^c + 5x_i^t + 30$$
$$t_i^t(x_i^c, x_i^t) = 1.3x_i^c + 2.6x_i^t + 28$$

Strict Monotonicty and Uniqueness

General Asymmetric TAP

Suppose that the link delay functions on link $i = 1, 2$ is given by

$$t_i^c(x_i^c, x_i^t) = 1.5x_i^c + 5x_i^t + 30$$
$$t_i^t(x_i^c, x_i^t) = 1.3x_i^c + 2.6x_i^t + 28$$

What are the class-specific travel times on the two links for the following flow solutions

$$\begin{pmatrix} x_1^c & x_2^c \\ x_1^t & x_2^t \end{pmatrix} = \begin{pmatrix} 4/3 & 44/3 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 8 & 8 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 44/3 & 4/3 \\ 0 & 4 \end{pmatrix}$$

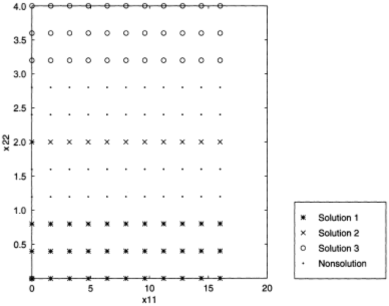
$$\begin{pmatrix} t_1^c & t_2^c \\ t_1^t & t_2^t \end{pmatrix} = \begin{pmatrix} 52 & 52 \\ 40 & 47 \end{pmatrix}, \begin{pmatrix} 52 & 52 \\ 44 & 44 \end{pmatrix}, \begin{pmatrix} 52 & 52 \\ 47 & 40 \end{pmatrix}$$

Do they satisfy the Wardrop principle?

Strict Monotonicty and Uniqueness

General Asymmetric TAP

Stability and Convergence of the Diagonalization algorithm from various starting points.



Marcotte, P., & Wynter, L. (2004). A new look at the multiclass network equilibrium problem. Transportation Science, 38(3), 282-292.

Lecture Outline

Optimization Model for Asymmetric TAP

The general approach to finding equilibria has been to find an equivalent convex program.

Can we still find an optimization model even if it is not convex?

Optimization Model for Asymmetric TAP

Gap Functions

Definition

Let X represent the feasible region of link flows. Define a gap function $g(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ as $\max_{\mathbf{x}' \in X} \mathbf{t}(\mathbf{x})^T(\mathbf{x} - \mathbf{x}')$

Does the gap function resemble something that we have seen so far?

Proposition

The VI($\mathbf{t}, X$) is equivalent to $\min g(\mathbf{x})$, *s.t.* $\mathbf{x} \in X$

Proof.

Notice that $g(\mathbf{x}) \geq 0 \, \forall \, \mathbf{x} \in X$ and $g(\mathbf{x}^*) = 0$ if $\mathbf{x}^*$ satisfies VI($\mathbf{t}, \mathbf{x}$) ∎

Optimization Model for Asymmetric TAP

Gap Functions

The gap function formulation is a min-max type program but the objective is in general nondifferentiable.

This makes it less useful since gradient descent-type approaches can no longer be used to solve the problem.

Is there an alternate optimization program in which the objective is differentiable?

Optimization Model for Asymmetric TAP

Extending Norms and Projections

Before, formulating such an optimization model, we will first generalize the definitions of norms and projections.

In the discussion that follows let $\mathbf{G}$ represent a given symmetric positive definite matrix.

Definition

The G-norm in $\mathbb{R}^n$ is defined as $\|\mathbf{x}\|_{\mathbf{G}} = \sqrt{\mathbf{x}^T\mathbf{G}\mathbf{x}}$

The old definitions can be recovered by simply setting $\mathbf{G} = \mathbf{I}$, the identity matrix.

Optimization Model for Asymmetric TAP

Extending Norms and Projections

Definition

For each $\mathbf{x}^* \in \mathbb{R}^n \exists! \mathbf{y} \in X$ such that

$$\mathbf{y} = \arg\min_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{x}^*\|$$

$\mathbf{y}$ is called the projection of $\mathbf{x}^*$ on X and is denoted by $\text{proj}_X(\mathbf{x}^*)$.

Definition

For each $\mathbf{x}^* \in \mathbb{R}^n \exists! \mathbf{y} \in X$ such that

$$\mathbf{y} = \arg\min_{\mathbf{x} \in X} \|\mathbf{x} - \mathbf{x}^*\|_{\mathbf{G}}$$

$\mathbf{y}$ is called the projection of $\mathbf{x}^*$ on X and is denoted by $\text{proj}_{X,\mathbf{G}}(\mathbf{x}^*)$.

Proposition

Projection mappings are non-expansive, i.e.,
$\|proj_{X,\mathbf{G}}(\mathbf{x}) - proj_{X,\mathbf{G}}(\mathbf{x}')\|_{\mathbf{G}} \leq \|\mathbf{x} - \mathbf{x}'\|_{\mathbf{G}} \, \forall \, \mathbf{x}, \mathbf{x}' \in X$

In other words, the G-distance between two points is always atleast greater than the G-distance between their projections.

Optimization Model for Asymmetric TAP

Extending Norms and Projections

Earlier, we saw that

Proposition

$\mathbf{y} = proj_X(\mathbf{x}^*) \Leftrightarrow (\mathbf{y} - \mathbf{x}^*)^T(\mathbf{x} - \mathbf{y}) \geq 0 \, \forall \, \mathbf{x} \in X$

Proposition

Suppose X is closed and convex. $\mathbf{x}^*$ is a solution to $VI(\mathbf{t}, X)$ iff $\mathbf{x}^*$ is a fixed point of $proj_X(\mathbf{x} - \mathbf{t}(\mathbf{x}))$, i.e., $\mathbf{x}^* = proj_X(\mathbf{x}^* - \mathbf{t}(\mathbf{x}^*))$

Similarly,

Proposition

$\mathbf{y} = proj_{X,\mathbf{G}}(\mathbf{x}^*) \Leftrightarrow (\mathbf{y} - \mathbf{x}^*)^T\mathbf{G}(\mathbf{x} - \mathbf{y}) \geq 0 \, \forall \, \mathbf{x} \in X$

Proposition

Suppose X is closed and convex. $\mathbf{x}^*$ is a solution to $VI(\mathbf{t}, X)$ iff $\mathbf{x}^*$ is a fixed point of $proj_{X,\mathbf{G}}(\mathbf{x} - \mathbf{G}^{-1}\mathbf{t}(\mathbf{x}))$, i.e., $\mathbf{x}^* = proj_{X,\mathbf{G}}(\mathbf{x}^* - \mathbf{G}^{-1}\mathbf{t}(\mathbf{x}^*))$

Optimization Model for Asymmetric TAP

Non-linear Program

Let $\mathbf{h}(\mathbf{x}) = \text{proj}_{X,\mathbf{G}}(\mathbf{x} - \mathbf{G}^{-1}\mathbf{t}(\mathbf{x}))$

Proposition

The VI($\mathbf{t}, X$) is equivalent to solving

$$\min_{\mathbf{x}\in X} f(\mathbf{x}) = -\mathbf{t}(\mathbf{x})^T(\mathbf{h}(\mathbf{x}) - \mathbf{x}) - \frac{1}{2}(\mathbf{h}(\mathbf{x}) - \mathbf{x})^T\mathbf{G}(\mathbf{h}(\mathbf{x}) - \mathbf{x})$$

Proof.

The objective can be re-written as

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2}\left\{\|\mathbf{G}^{-1}\mathbf{t}(\mathbf{x})\|_{\mathbf{G}}^2 - \|\mathbf{h}(\mathbf{x}) - (\mathbf{x} - \mathbf{G}^{-1}\mathbf{t}(\mathbf{x}))\|_{\mathbf{G}}^2\right\} \\ &= \frac{1}{2}\left\{\|\mathbf{x} - (\mathbf{x} - \mathbf{G}^{-1}\mathbf{t}(\mathbf{x}))\|_{\mathbf{G}}^2 - \|\mathbf{h}(\mathbf{x}) - (\mathbf{x} - \mathbf{G}^{-1}\mathbf{t}(\mathbf{x}))\|_{\mathbf{G}}^2\right\} \geq 0 \end{aligned}$$

Further, $f(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{h}(\mathbf{x})$. But if $\mathbf{x} = \mathbf{h}(\mathbf{x})$, then $\mathbf{x}$ solves VI($\mathbf{t}, X$). ■

Optimization Model for Asymmetric TAP

Gradient of the Objective

It can also be shown that the objective is differentiable and its gradient is

$$\nabla f(\mathbf{x}) = \mathbf{t}(\mathbf{x}) - (\nabla \mathbf{t}(\mathbf{x}) - \mathbf{G})(\mathbf{h}(\mathbf{x}) - \mathbf{x})$$

However, the problem is non-convex and hence local optima may exist!

Optimization Model for Asymmetric TAP

Special Cases

But if the Jacobian $\mathbf{t}(\mathbf{x})$ is positive definite, i.e., the travel time functions are strictly monotone, then it can be shown that any stationary point is a global optimum.

In this case, the vector $\mathbf{h}(\mathbf{x}) - \mathbf{x}$ is a descent direction and hence one need not even evaluate the Jacobian matrix while computing the optimum solution.

Historic Notes

Supplementary Reading

Dafermos, S. (1982). Relaxation algorithms for the general asymmetric traffic equilibrium problem. Transportation Science, 16(2), 231-240.

Fukushima, M. (1992). Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. Mathematical programming, 53(1-3), 99-110.

Your Moment of Zen