CE 272
Traffic Network Equilibrium

Lecture 13

Multi-Class User Equilibrium - Part I

Previously on Traffic Network Equilibrium...

The demand information for all OD pairs is commonly referred to as **OD matrix** or **trip tables**.

The number of person trips are computed from the first two steps of the four-step process. In the third step, these trips are assigned to different modes (car, bus, two-wheeler etc.) resulting in a trip table for each mode.

But for equilibrium analysis, we assume that demand comprises of only passenger cars. The demand of other types of vehicles are adjusted by factors called **passenger car units (PCUs)** that reflect their sizes relative to that of a car.

Previously on Traffic Network Equilibrium...

So far, we have established that

1. If $\mathbf{t}(\mathbf{x})$ is continuous, the function $\text{proj}_X(\mathbf{x} - \mathbf{t}(\mathbf{x}))$ has fixed points.
2. These fixed points solve $\text{VI}(\mathbf{t}, X)$.

The last piece of the puzzle is to prove that the solutions to the VI are actually Wardrop equilibria.

Previously on Traffic Network Equilibrium...

Theorem

$\mathbf{x}^*$ *satisfies the* $VI(\mathbf{t}, X) \Leftrightarrow$ *it satisfies the Wardrop principle*

Proof.

($\Rightarrow$) Since $\mathbf{x}^*$ satisfies the VI, $\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0$, i.e,

$$\mathbf{t}(\mathbf{x}^*)^T\mathbf{x}^* \leq \mathbf{t}(\mathbf{x}^*)^T\mathbf{x}\,\forall\,\mathbf{x} \in X$$

Imagine the path travel times are fixed at $\mathbf{t}(\mathbf{x}^*)$. The RHS, $\mathbf{t}(\mathbf{x}^*)^T\mathbf{x}$ is the total system travel time (TSTT) incurred by the flow pattern $\mathbf{x}$.

When the path travel times are fixed, TSTT is minimized if we route travelers on least travel time paths between each OD pair. Thus, from the above inequality $\mathbf{x}^*$ is a Wardrop equilibrium.

($\Leftarrow$) Exercise. ∎

Previously on Traffic Network Equilibrium...

The objective is convex in η. (Why?) So the first-order conditions imply that $f'(\eta) = 0$ at an interior η.

$$\frac{d}{d\eta} \sum_{(i,j)\in A} \int_0^{\eta \hat{x}_{ij} + (1-\eta)x_{ij}} t_{ij}(\omega)\, d\omega = 0$$

$$\Rightarrow \sum_{(i,j)\in A} t_{ij}\left(\eta \hat{x}_{ij} + (1-\eta)x_{ij}\right)\left(\hat{x}_{ij} - x_{ij}\right) = 0$$

Solving this equation provides the optimal η (assuming it lies in the interior) which then gives the next iterate.

Within each FW iteration, more computations are needed compared to MSA but the overall number of iterations are reduced.

Lecture Outline

1. Multi-Class User Equilibrium
2. Symmetric Traffic Assignment
3. Asymmetric Traffic Assignment
4. Diagonalization

Lecture Outline

Multi-Class User Equilibrium

Earlier, we assumed that traffic was homogeneous and converted the demand for different modes into PCUs.

- Can vehicle composition on a link affect link travel times for different modes?
- Is it possible to model equilibrium flows for different vehicle types?

Imagine a network with two types of vehicles: cars and two-wheelers. Suppose the flows of cars is x_{ij}^c and x_{ij}^t.

Let the link delays for cars be $t_{ij}^c(x_{ij}^c, x_{ij}^t)$ and that for two-wheelers be $t_{ij}^t(x_{ij}^c, x_{ij}^t)$.

As before, we first try to find a convex programming formulation that gives the equilibrium $\mathbf{x}^c$ and $\mathbf{x}^t$.

Symmetric Traffic Assignment

Symmetric Traffic Assignment

Assumptions

Unfortunately, it is not possible to formulate a convex program except when

$$\frac{\partial t^c_{ij}(x^c_{ij}, x^t_{ij})}{\partial x^t_{ij}} = \frac{\partial t^t_{ij}(x^c_{ij}, x^t_{ij})}{\partial x^c_{ij}} \, \forall \, (i, j) \in A$$

In words, the impact of an additional two-wheeler on the travel time of cars is same as the impact of an additional car on the travel time of two-wheelers.

The assumption is called symmetric assumption. Do the following BPR-type functions satisfy this condition?

$$t^c_{ij}(x^c_{ij}, x^t_{ij}) = t^0_{ij}\left(1 + \alpha\left(\frac{x^c_{ij} + 2x^t_{ij}}{2000}\right)^{\beta}\right)$$

$$t^t_{ij}(x^c_{ij}, x^t_{ij}) = t^0_{ij}\left(1 + \alpha\left(\frac{2x^c_{ij} + x^t_{ij}}{2000}\right)^{\beta}\right)$$

More generally, multi-class equilibrium falls under a class of problems in which link delays are no longer separable and can be written as $\mathbf{t}(\mathbf{x})$.

The Jacobian of a vector-valued function $\mathbf{t}(\mathbf{x})$ is defined as

$$\begin{bmatrix} \frac{\partial t_1}{\partial x_1} & \frac{\partial t_1}{\partial x_2} & \cdots & \frac{\partial t_1}{\partial x_m} \\ \frac{\partial t_2}{\partial x_1} & \frac{\partial t_2}{\partial x_2} & \cdots & \frac{\partial t_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_m}{\partial x_1} & \frac{\partial t_m}{\partial x_2} & \cdots & \frac{\partial t_m}{\partial x_m} \end{bmatrix}$$

Under the earlier assumption, the Jacobian $\nabla\mathbf{t}(\mathbf{x})$ is a symmetric matrix and hence this class of problems are also called symmetric TAPs.

We say that a solution $(\mathbf{x}^c, \mathbf{x}^t)$ is an Wardrop equilibrium if for each mode, all used paths have equal and minimal travel times.

Symmetric Traffic Assignment

Convex Program

The multi-class equilibrium problem for symmetric delay functions can be solved using

$$\min \frac{1}{2} \sum_{(i,j)\in A} \left(\int_0^{x_{ij}^c} t_{ij}^c(\omega, x_{ij}^t)\, d\omega + \int_0^{x_{ij}^c} t_{ij}^c(\omega, 0)\, d\omega \right.$$
$$\left. + \int_0^{x_{ij}^t} t_{ij}^t(x_{ij}^c, \omega)\, d\omega + \int_0^{x_{ij}^t} t_{ij}^t(0, \omega)\, d\omega \right)$$

$$\begin{aligned} \text{s.t.} \quad \sum_{p\in P_{rs}} y_p^c &= d_{rs}^c \ \forall\ (r,s) \in Z^2 \\ \sum_{p\in P_{rs}} y_p^t &= d_{rs}^t \ \forall\ (r,s) \in Z^2 \\ y_p^c &\geq 0 \ \forall\ p \in P \\ y_p^t &\geq 0 \ \forall\ p \in P \end{aligned}$$

Symmetric Traffic Assignment

Convex Program

$$\mathcal{L}(\mathbf{y}, \boldsymbol{\lambda}^c, \boldsymbol{\lambda}^t, \boldsymbol{\mu}^c, \boldsymbol{\mu}^t) = \frac{1}{2} \sum_{(i,j)\in A} \left(\int_0^{x_{ij}^c} t_{ij}^c(\omega, x_{ij}^t)\, d\omega + \int_0^{x_{ij}^c} t_{ij}^c(\omega, 0)\, d\omega \right.$$
$$\left. + \int_0^{x_{ij}^t} t_{ij}^t(x_{ij}^c, \omega)\, d\omega + \int_0^{x_{ij}^t} t_{ij}^t(0, \omega)\, d\omega \right) + \sum_{p\in P} \lambda_p^c(-y_p^c) + \sum_{p\in P} \lambda_p^t(-y_p^t)$$
$$+ \sum_{(r,s)\in Z^2} \mu_{rs}^c \left(d_{rs}^c - \sum_{p\in P_{rs}} y_p^c \right) + \sum_{(r,s)\in Z^2} \mu_{rs}^t \left(d_{rs}^t - \sum_{p\in P_{rs}} y_p^t \right)$$

Symmetric Traffic Assignment

Convex Program

Primal feasibility:

$$\sum_{p \in P_{rs}} y_p^c = d_{rs}, \sum_{p \in P_{rs}} y_p^t = d_{rs} \; \forall \; (r, s) \in Z^2$$

$$y_p^c \geq 0, y_p^t \geq 0 \; \forall \; p \in P$$

Dual feasibility:

$$\lambda_p^c \geq 0, \lambda_p^t \geq 0 \, \forall \, p \in P$$

Complementary Slackness:

$$\lambda_p^c y_p^c = 0, \lambda_p^t y_p^t = 0 \, \forall \, p \in P$$

Gradient of the Lagrangian vanishes:

$$\sum_{(i,j) \in A} \delta_{ij}^p t_{ij}^c(x_{ij}^c, x_{ij}^t) - \lambda_p^c - \mu_{rs}^c = 0 \, \forall \, (r, s) \in Z^2, p \in P_{rs}$$

$$\sum_{(i,j) \in A} \delta_{ij}^p t_{ij}^t(x_{ij}^c, x_{ij}^t) - \lambda_p^t - \mu_{rs}^t = 0 \, \forall \, (r, s) \in Z^2, p \in P_{rs}$$

Symmetric Traffic Assignment

Convex Program

From the last three conditions, eliminating λ_p^c, for all $(r,s) \in Z^2$, $p \in P_{rs}$,

$$\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}^c(x_{ij}^c, x_{ij}^t) \geq \mu_{rs}^c$$

$$y_p^c \left(\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}^c(x_{ij}^c, x_{ij}^t) - \mu_{rs}^c \right) = 0$$

Similarly, for two-wheelers,

$$\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}^t(x_{ij}^c, x_{ij}^t) \geq \mu_{rs}^t$$

$$y_p^t \left(\sum_{(i,j)\in A} \delta_{ij}^p t_{ij}^t(x_{ij}^c, x_{ij}^t) - \mu_{rs}^t \right) = 0$$

These are essentially conditions for Wardrop equilibrium.

Symmetric Traffic Assignment

Solution Methods

The earlier convex optimization model was a simple extension of the Beckmann formulation.

Hence, both MSA and FW can be adapted to find the optimal solution.

Lecture Outline

Asymmetric Traffic Assignment

The symmetric assumption is too strong since the flows for different modes have differential effects on the travel times of other modes.

In other words, it is very likely that

$$\frac{\partial t_{ij}^{c}(x_{ij}^{c}, x_{ij}^{t})}{\partial x_{ij}^{t}} \neq \frac{\partial t_{ij}^{t}(x_{ij}^{c}, x_{ij}^{t})}{\partial x_{ij}^{c}} \, \forall \, (i,j) \in A$$

Such equilibrium problems are also called asymmetric TAPs and do not impose any structure on the Jacobian of the travel times.

Unfortunately, it is not possible to write a simple convex programming model for such instances.

Asymmetric Traffic Assignment

Road Map

For asymmetric TAPs, we will have to use VIs to model equilibrium flows. We have seen that VIs have a solution when $\mathbf{t}(\mathbf{x})$ is continuous, but we haven't discussed

- Uniqueness of VI solutions
- Algorithmic procedures to compute equilibria

We will first address these aspects by making an additional assumption called **strict monotonicity** and then study a more relaxed version of the problem.

Definition

The travel time mapping $\mathbf{t}(\mathbf{x}) : X \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^m$ is said to be *strictly monotone* if

$$\big(\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x}')\big)^T (\mathbf{x} - \mathbf{x}') > 0 \, \forall \, \mathbf{x}, \mathbf{x}' \in X, \mathbf{x} \neq \mathbf{x}'$$

If delay functions are strictly increasing in their arguments, it does not imply that they are strictly monotone.

For example, consider $t_1(x_1, x_2) = x_1 + 2x_2$ and $t_2(x_1, x_2) = 2x_1 + x_2$. Evaluate the dot product at $x_1 = 0, x_2 = 6$ and $x_1' = 5, x_2' = 1$.

Loosely speaking, strict monotonicity implies that the diagonal terms of the Jacobian are large compared to the off-diagonal terms.

Asymmetric Traffic Assignment

Strict Monotonicity

Proposition

Let $\mathbf{t}(\mathbf{x})$ be a continuous function on a convex domain. $\mathbf{t}$ is strictly monotone $\Leftrightarrow$ its Jacobian is positive definite.

Proof.

($\Leftarrow$) Using the mean value theorem, $t_{ij}(\mathbf{x}) - t_{ij}(\mathbf{x}') = \nabla t_{ij}(\mathbf{z})^T(\mathbf{x}' - \mathbf{x})$ where $\mathbf{z} = \mathbf{x} + \omega(\mathbf{x} - \mathbf{x}')$

$$\Rightarrow (x_{ij} - x'_{ij})(t_{ij}(\mathbf{x}) - t_{ij}(\mathbf{x}')) = (x_{ij} - x'_{ij})\nabla t_{ij}(\mathbf{z})^T(\mathbf{x} - \mathbf{x}')$$

$$\Rightarrow \sum_{(i,j)\in A} (x_{ij} - x'_{ij})(t_{ij}(\mathbf{x}) - t_{ij}(\mathbf{x}')) = \sum_{(i,j)\in A} (x_{ij} - x'_{ij})\nabla t_{ij}(\mathbf{z})^T(\mathbf{x} - \mathbf{x}')$$

$$\Rightarrow (\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x}'))^T(\mathbf{x} - \mathbf{x}') = \sum_{(i,j)\in A} (x_{ij} - x'_{ij})\nabla t_{ij}(\mathbf{z})^T(\mathbf{x} - \mathbf{x}')$$

$$\Rightarrow (\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x}'))^T(\mathbf{x} - \mathbf{x}') = (\mathbf{x} - \mathbf{x}')^T\nabla \mathbf{t}(\mathbf{z})(\mathbf{x} - \mathbf{x}') > 0$$

($\Rightarrow$) Exercise. ∎

Asymmetric Traffic Assignment

Proposition

If $\mathbf{t}$ *is strictly monotone and continuous, then a solution to the VI(*$\mathbf{t}, X$*) is unique.*

Proof.

Suppose not. Let $\mathbf{x}^*$ and $\mathbf{x}'$ be two solutions that satisfy the VI($\mathbf{t}, X$). Since both flows satisfy the VI,

$$\mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0 \, \mathbf{x} \in X$$
$$\mathbf{t}(\mathbf{x}')^T(\mathbf{x} - \mathbf{x}') \geq 0 \, \mathbf{x} \in X$$

The above two inequalities imply

$$(\mathbf{t}(\mathbf{x}^*) - \mathbf{t}(\mathbf{x}'))^T(\mathbf{x}' - \mathbf{x}^*) \geq 0$$
$$\Rightarrow (\mathbf{t}(\mathbf{x}^*) - \mathbf{t}(\mathbf{x}'))^T(\mathbf{x}^* - \mathbf{x}') \leq 0$$

which contradicts that $\mathbf{t}$ is strictly monotone. ■

Diagonalization

Diagonalization

Introduction

Diagonalization is a FW-like method to solve asymmetric equilibrium problems. Assume that the travel time functions are strictly monotone.

For a fixed set of link flows, the delay functions can be used to get the shortest paths and the all-or-nothing flows.

But since we do not have a convex programming objective, we cannot pick a step size that minimizes the Beckmann function.

Diagonalization

Method

At each iteration, select an arc (i,j). Fix the flows of all the other arcs and construct real-valued functions for the delay all the arcs in the network.

Suppose the flow at the kth iteration is $(x_1^k, x_2^k, \ldots, x_m^k)$. Solve the sub-problem in which the objective is

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(x_1^k, x_2^k, \ldots, \omega, \ldots x_m^k)\, d\omega$$

Let $\tilde{t}_{ij}(x_{ij}) = t_{ij}(x_1^k, x_2^k, \ldots, x_{ij}, \ldots x_m^k)$. Then, the step size η in the FW iteration is obtained by solving

$$\sum_{(i,j)\in A} \tilde{t}_{ij}(\eta \hat{x}_{ij} + (1-\eta)x_{ij})(\hat{x}_{ij} - x_{ij}) = 0$$

Diagonalization

Method

The method is called diagonalization since the Jacobian of each of the sub-problems is a diagonal matrix.

Your Moment of Zen