CE 272
Traffic Network Equilibrium

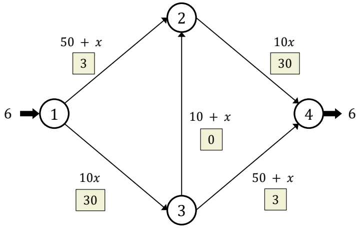
Lecture 12

Traffic Assignment with Elastic Demand - Part II

Previously on Traffic Network Equilibrium...

First-best tolls for achieving SO flows are not unique. Hence, one can seek tolls that satisfy some secondary objective.

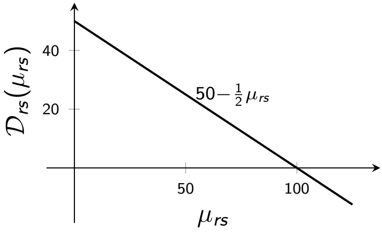
What are the UE flows in the following networks?

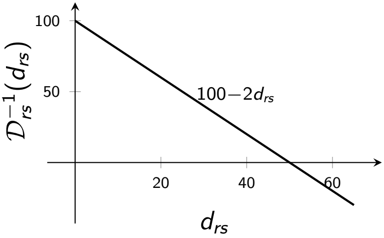


What is the total revenue in both cases? Why might we want to collect minimum revenue tolls?

Previously on Traffic Network Equilibrium...

Consider the demand function $\mathcal{D}_{rs}(\mu_{rs}) = 50 - \frac{1}{2}\mu_{rs}$ and its inverse $\mathcal{D}_{rs}^{-1}(d_{rs}) = 100 - 2d_{rs}$





We will let the demand function $\mathcal{D}$ take negative values so that its inverse exists. Negative demand values are avoid by setting $d_{rs} = \mathcal{D}^{+}(\mu_{rs}) = \max\{\mathcal{D}(\mu_{rs}), 0\}$

Previously on Traffic Network Equilibrium...

The equilibrium solution for the elastic demand problem can be obtained by solving the following convex program.

$$\min_{\mathbf{y},\mathbf{d}} \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega)\, d\omega$$

$$\text{s.t.} \sum_{p\in P_{rs}} y_p = d_{rs} \;\forall\; (r,s) \in Z^2$$

$$y_p \geq 0 \;\forall\; p \in P$$

$$d_{rs} \geq 0 \;\forall\; (r,s) \in Z^2$$

Note that both terms in the objective have units of time.

Previously on Traffic Network Equilibrium...

As before, eliminating λ_p, for all $(r, s) \in Z^2$, $p \in P_{rs}$, we get

$$\tau_p(\mathbf{y}) \geq \mu_{rs}$$
$$y_p \left(\tau_p(\mathbf{y}) - \mu_{rs}\right) = 0$$

These two imply the Wardrop principle. Eliminating ν_{rs}, for all $(r, s) \in Z^2$

$$\mu_{rs} \geq \mathcal{D}_{rs}^{-1}(d_{rs})$$
$$d_{rs} \left(\mathcal{D}_{rs}^{-1}(d_{rs}) - \mu_{rs}\right) = 0$$

Hence, if the demand between an OD pair is strictly positive $\mathcal{D}_{rs}^{-1}(d_{rs}) = \mu_{rs}$. Else, if it zero, the shortest path time is greater than or equal to time value at which no users are willing to travel between O and D (i.e., $\mathcal{D}_{rs}^{-1}(0)$).

Lecture Outline

1. Example
2. Gartner Transformation
3. System Optimum with Elastic Demand

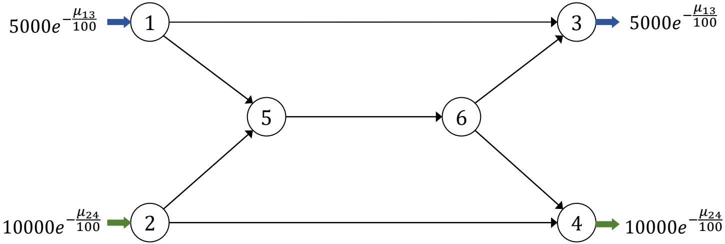
Lecture Outline

Example

Example

Example 1

Find the equilibrium link flows and demand using MSA and FW



Example

Method of Successive Averages

MSA(G)

$k \leftarrow 1$
Find a feasible $\hat{\mathbf{x}}$, $\hat{\mathbf{d}}$

while Relative Gap $> 10^{-4}$ or $TMF > \epsilon$ **do**
  $\mathbf{x} \leftarrow \frac{1}{k}\hat{\mathbf{x}} + (1 - \frac{1}{k})\mathbf{x}$
  $\mathbf{d} \leftarrow \frac{1}{k}\hat{\mathbf{d}} + (1 - \frac{1}{k})\mathbf{d}$
  Update $\mathbf{t}(\mathbf{x})$
  $\hat{\mathbf{x}} \leftarrow \mathbf{0}$
  for $r \in Z$ **do**
    Dijkstra (G, r)
    for $s \in Z, (i, j) \in p^*_{rs}$ **do**
      $\hat{d}_{rs} \leftarrow \mathcal{D}^+_{rs}(\mu^*_{rs})$
      $\hat{x}_{ij} \leftarrow \hat{x}_{ij} + \hat{d}_{rs}$
    end for
  end for
  $TMF \leftarrow \sum_{(r,s)\in Z^2} |\hat{d}_{rs} - d_{rs}|$
  Relative Gap $\leftarrow TSTT/SPTT - 1$
  $k \leftarrow k + 1$
end while

Example

Frank-Wolfe

FRANK-WOLFE(G)

```
k ← 1
Find a feasible x̂, d̂

while Relative Gap > 10^-4 or TMF > ε do
    if k = 1 then η ← 1 else η ← BISECTION(G, x, d, x̂, d̂)
    x ← ηx̂ + (1 − η)x
    d ← ηd̂ + (1 − η)d
    Update t(x)
    x̂ ← 0
    for r ∈ Z do
        DIJKSTRA (G, r)
        for s ∈ Z, (i, j) ∈ p*_rs do
            d̂_rs ← D+_rs(μ*_rs)
            x̂_ij ← x̂_ij + d̂_rs
        end for
    end for
    TMF ← Σ_{(r,s)∈Z²} |d̂_rs − d_rs|
    Relative Gap ← TSTT/SPTT − 1
    k ← k + 1
end while
```

Lecture Outline

Gartner Transformation

Gartner Transformation

Artificial Links

Demand functions have a maximum demand that represents the number of users who are likely to travel. Denote this by $\bar{d}_{rs}$.

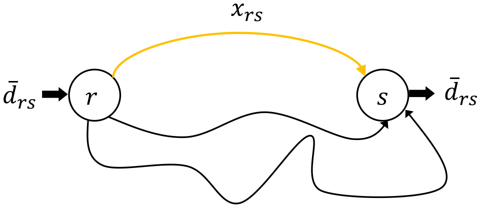
The idea behind Gartner transformation is to create an artificial link between each OD pair.

The artificial link will carry users who choose to not travel. The remaining demand will be routed along actual roadway links.

Gartner Transformation

Choosing Delay Functions

Let x_{rs} represent users who do not travel. Hence, $\bar{d}_{rs} - x_{rs} = d_{rs}$ is the number of users who actually travel.



What should the delay on the artificial links be? Gartner in 1980 suggested that the delay be set to $\mathcal{D}_{rs}^{-1}(\bar{d}_{rs} - x_{rs})$, where x_{rs} is the flow on the artificial link (r, s).

Gartner Transformation

Objective Function

The Beckmann function for a network with artificial links can be written as

$$\sum_{(i,j)\in A}\int_0^{x_{ij}} t_{ij}(\omega)\,d\omega + \sum_{(r,s)\in Z^2}\int_0^{x_{rs}} \mathcal{D}_{rs}^{-1}(\bar{d}_{rs}-\omega)\,d\omega$$

Let $\omega' = \bar{d}_{rs} - \omega$. Therefore, $d\omega' = -d\omega$ and the objective becomes

$$\sum_{(i,j)\in A}\int_0^{x_{ij}} t_{ij}(\omega)\,d\omega - \sum_{(r,s)\in Z^2}\int_{\bar{d}_{rs}}^{\bar{d}_{rs}-x_{rs}} \mathcal{D}_{rs}^{-1}(\omega')\,d\omega'$$

$$= \sum_{(i,j)\in A}\int_0^{x_{ij}} t_{ij}(\omega)\,d\omega - \sum_{(r,s)\in Z^2}\left(\int_{\bar{d}_{rs}}^{0} \mathcal{D}_{rs}^{-1}(\omega')\,d\omega' + \int_0^{\bar{d}_{rs}-x_{rs}} \mathcal{D}_{rs}^{-1}(\omega')\,d\omega'\right)$$

$$= \sum_{(i,j)\in A}\int_0^{x_{ij}} t_{ij}(\omega)\,d\omega - \sum_{(r,s)\in Z^2}\int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega')\,d\omega' + \text{Constant}$$

Gartner Transformation

VI Version

The elastic demand problem can also be expressed as a VI. Let D denote the set of feasible demand vectors, i.e., $D = \{\mathbf{d} : d_{rs} \leq \bar{d}_{rs}, d_{rs} \geq 0 \, \forall \, (r, s) \in Z^2\}$.

Let X a feasible set of link flows, which contains vectors $\mathbf{x}$ that satisfy,

$$\sum_{p \in P_{rs}} y_p = d_{rs} \, \forall \, (r, s) \in Z^2, \mathbf{d} \in D$$

$$x_{ij} = \sum_{p \in P} \delta_{ij}^p y_p \, \forall \, (i, j) \in A$$

$$y_p \geq 0 \, \forall \, p \in P$$

Proposition

$(\mathbf{x}^*, \mathbf{d}^*)$ *is an equilibrium* $\Leftrightarrow \mathbf{t}(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) - (\mathcal{D}^{-1}(\mathbf{d}^*))^T(\mathbf{d} - \mathbf{d}^*) \geq 0$ $\forall \, \mathbf{x}, \mathbf{d} \in X \times D$

Lecture Outline

System Optimum with Elastic Demand

System Optimum with Elastic Demand

Defining the Objective

To find a system optimum solution for the elastic demand problem, one option is to minimize the TSTT as before.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) \\
\text{s.t.} \quad & \sum_{p\in P_{rs}} y_p = d_{rs} \;\forall\; (r,s) \in Z^2 \\
& x_{ij} = \sum_{p\in P} \delta_{ij}^{p} y_p \,\forall\, (i,j) \in A \\
& y_p \geq 0 \;\forall\; p \in P \\
& d_{rs} \geq 0 \;\forall\; (r,s) \in Z^2
\end{aligned}
$$

What's wrong with this approach?

System Optimum with Elastic Demand

Defining the Objective

- The demand/inverse demand functions do not appear in this model, so elastic demand is not being modeled. In fact, setting d_{rs} to 0 is optimal!
- Likewise, if we imagine a SO state is induced by tolls, one could set very high tolls that would prevent everyone from traveling and thus minimize TSTT.

Minimizing TSTT is clearly not the right objective. To get the complementary slackness conditions, we need a second term that involves the inverse demand function. Thus, the objective of the SO problem is

$$\sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega) d\omega$$

System Optimum with Elastic Demand

Consumer Surplus

How do we interpret the objective? Imagine a single link between each OD pair.

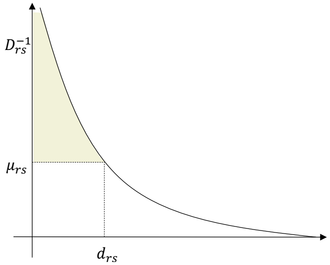
$$\begin{aligned}
&\sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij}) - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega) d\omega \\
&= \sum_{(r,s)\in Z^2} \left(d_{rs}\mu_{rs} - \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega) d\omega \right) \\
&= - \sum_{(r,s)\in Z^2} \left(\int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega) d\omega - d_{rs}\mu_{rs} \right) \\
&= -CS
\end{aligned}$$

where CS denotes the consumer surplus.

System Optimum with Elastic Demand

Consumer Surplus

Graphically, $CS = \sum_{(r,s)\in Z^2} \left(\int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega) d\omega - d_{rs}\mu_{rs} \right)$ can be represented using the shaded portion.



One way to interpret the demand functions is to assume that each traveler has a threshold time below which he or she is willing to travel.

Thus, $\mathcal{D}_{rs}(\mu_{rs})$ represents all travelers who have a threshold greater than or equal to μ_{rs}.

Suppose a traveler's threshold is 15 min but it takes 10 min to finish the trip. Hence, the traveler 'saves' 5 min and aggregating these values across all travelers gives the consumer surplus.

System Optimum with Elastic Demand

Consumer Surplus

In economics, consumer surplus is defined as the difference between the price individuals are willing to pay for a good and the actual purchase price.

It is a measure of benefit to users and maximizing it is thus equivalent to maximizing social welfare.

How do you measure CS in traffic networks? Can you think of markets where CS is easier to measure?

http://freakonomics.com/podcast/uber-economists-dream/

Cohen, P., Hahn, R., Hall, J., Levitt, S., & Metcalfe, R. (2016). Using big data to estimate consumer surplus: The case of Uber (No. w22627). National Bureau of Economic Research. [PDF]

Thus, the SO problem can be obtained by solving the following optimization program.

$$\min \; -CS = \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega)d\omega - \sum_{(i,j)\in A} x_{ij} t_{ij}(x_{ij})$$

$$\text{s.t.} \; \sum_{p\in P_{rs}} y_p = d_{rs} \; \forall \; (r,s) \in Z^2$$

$$x_{ij} = \sum_{p\in P} \delta_{ij}^p y_p \, \forall \, (i,j) \in A$$

$$y_p \geq 0 \; \forall \; p \in P$$

$$d_{rs} \geq 0 \; \forall \; (r,s) \in Z^2$$

As before, we can replace the link delays with the marginal cost functions $\hat{t}_{ij}(x) = t_{ij}(x) + x t'_{ij}(x)$ and solve the UE problem to get the SO flows.

System Optimum with Elastic Demand

Marginal Cost Tolling

We can also achieve a SO by setting a toll of $x_{ij}^{SO} t'_{ij}(x_{ij}^{SO})$ on each link and by solving the UE problem, assuming that drivers behave selfishly.

When demand is inelastic, the set of link tolls that induce a SO is not unique. This is also true for the elastic demand case.

However, all SO tolls generate the same revenue!

System Optimum with Elastic Demand

Revenue Analysis

To see why, consider the UE problem with a toll of c_{ij} on link (i, j).

$$\min_{\mathbf{y},\mathbf{d}} \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} (t_{ij}(\omega) + c_{ij})\, d\omega - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega)\, d\omega$$

$$\text{s.t.} \sum_{p\in P_{rs}} y_p = d_{rs} \ \forall\ (r,s) \in Z^2$$

$$y_p \geq 0 \ \forall\ p \in P$$

$$d_{rs} \geq 0 \ \forall\ (r,s) \in Z^2$$

System Optimum with Elastic Demand

Revenue Analysis

Recall that the KKT conditions without tolls were

$$\tau_p(\mathbf{y}) \geq \mu_{rs}$$
$$y_p\left(\tau_p(\mathbf{y}) - \mu_{rs}\right) = 0$$
$$\mu_{rs} \geq \mathcal{D}_{rs}^{-1}(d_{rs})$$
$$d_{rs}\left(\mathcal{D}_{rs}^{-1}(d_{rs}) - \mu_{rs}\right) = 0$$

In the presence of tolls, the second equation takes the form

$$y_p\left(\tau_p(\mathbf{y}) + \sum_{(i,j)\in A} \delta_{ij}^p c_{ij} - \mu_{rs}\right) = 0 \,\forall\, (r,s) \in Z^2, p \in P_{rs}$$

System Optimum with Elastic Demand

Revenue Analysis

For tolls that induce SO flows and demands,

$$d_{rs}^{SO}\left(\mathcal{D}_{rs}^{-1}(d_{rs}^{SO}) - \mu_{rs}\right) = 0\,\forall\,(r,s) \in Z^2$$

$$y_p^{SO}\left(\tau_p(\mathbf{y}^{SO}) + \sum_{(i,j)\in A} \delta_{ij}^p c_{ij} - \mu_{rs}\right) = 0\,\forall\,(r,s) \in Z^2, p \in P_{rs}$$

Adding these across all OD pairs, and subtracting one from the other,

$$\sum_{(r,s)\in Z^2}\sum_{p\in P_{rs}} y_p^{SO}\left(\tau_p(\mathbf{y}^{SO}) + \sum_{(i,j)\in A} \delta_{ij}^p c_{ij} - \mu_{rs}\right) - \sum_{(r,s)\in Z^2} d_{rs}^{SO}\left(\mathcal{D}_{rs}^{-1}(d_{rs}^{SO}) - \mu_{rs}\right) = 0$$

Therefore,

$$\sum_{p\in P} y_p^{SO} \sum_{(i,j)\in A} \delta_{ij}^p c_{ij} = \sum_{(r,s)\in Z^2} d_{rs}^{SO}\mathcal{D}_{rs}^{-1}(d_{rs}^{SO}) - \sum_{p\in P} y_p^{SO}\tau_p(\mathbf{y}^{SO})$$

Assuming that the objective is strictly convex, the SO solution is unique and hence the RHS is a constant.

Your Moment of Zen