CE 272
Traffic Network Equilibrium

Lecture 10

Connections with Potential Games

Previously on Traffic Network Equilibrium...

Nash Equilibrium (1951)

At equilibrium, no player has an incentive to deviate.

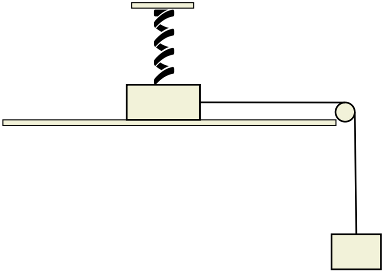
	Do nothing	Detonate
Do nothing	0, 0	-3, $+1$
Detonate	$+1$, -3	-2, -2

To check if a particular outcome is an equilibrium solution

1. Pick a player i
2. Fix the actions of the other players that lead to the current outcome
3. Check if player i can do better by selecting a different action
4. Repeat steps 1-3 for all players

Previously on Traffic Network Equilibrium...

One can write an expression for the energy of the system and minimize it.



We'll follow a similar line of thought for computing traffic equilibria because it can scale up well.

Previously on Traffic Network Equilibrium...

Can we reverse engineer a convex function such that the KKT conditions are equivalent to the Wardrop equilibria?

Martin Beckmann, C. B. McGuire, and Christopher Winsten in 1956 discovered such a function in their seminal book *Studies in the Economics of Transportation*.

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

This function is commonly referred to as the Beckmann function.

Previously on Traffic Network Equilibrium...

We can rewrite the earlier formulation purely in terms of the link flow variables, i.e., the decision variables are the xs.

$$\min \sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in A} x_{ij}^{rs} - \sum_{h:(h,i)\in A} x_{hi}^{rs} = \begin{cases} d_{is} & \text{if } i = r \\ -d_{ri} & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \quad \forall\,(r,s) \in Z^2$$

$$x_{ij} = \sum_{(r,s)\in Z^2} x_{ij}^{rs}\,\forall\,(i,j) \in A$$

$$x_{ij}^{rs} \geq 0\,\forall\,(i,j) \in A, (r,s) \in Z^2$$

This optimization program, also called the Beckmann formulation, has fewer variables and is easier to solve.

Lecture Outline

Mixed Strategy Equilibria

Mixed Strategy Equilibria

Introduction

Find the equilibrium in the following game

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

None of the four outcomes in this game (also called matching pennies) is an equilibrium. These type of outcomes are called pure strategy equilibira. You can imagine this game to be similar to how a goalkeeper and penalty taker in football may behave.

But the game has an equilibrium if we allow players to randomize.

Mixed Strategy Equilibria

Computing Mixed Equilibria

Such randomized strategies are called mixed strategy equilibria. Suppose the row player chooses H or T with probabilities p and $1-p$. Assume that the column player picks H or T with probabilities q and $1-q$.

	H	T
H	1,−1	−1,1
T	−1,1	1,−1

Cosider the row player.
Expected utility of selecting $H = q(1) + (1-q)(-1) = 2q - 1$
Expected utility of selecting $T = q(-1) + (1-q)(1) = 1 - 2q$

If the row player randomizes, these two must be equal. (Why?)
Therefore, $2q - 1 = 1 - 2q \Rightarrow q = \frac{1}{2}$. Similarly, $p = \frac{1}{2}$.

Compute the pure or mixed strategy equilibria in the following game of Rock-Paper-Scissors.

	R	P	S
R	0,0	−1,1	1,−1
P	1,−1	0,0	−1,1
S	−1,1	1,−1	0,0

Mixed Strategy Equilibria

Notation

Just for this lecture, we will use notation that's different from what was used so far. These descriptors are more common in the game theory literature. We will also assume players maximize utilities.

Notation	Description
Γ	A Normal form game
N	Set of Players (*Travelers in the network*)
$a_i \in A_i$	Pure strategies for player i (*Set of all available routes*)
$s_i(a_i)$	The probability with which player i chooses an action a_i
S_i	Set of all mixed strategies for player i
$u_i(s)$	Utility functions (*Travel time for a given strategy*)
$-i$	Opponents of player i
$BR_i(s)$	Best responses for player i (*Set of shortest paths*)

Mixed Strategy Equilibria

Definition

Definition

Given $s \in S$, the best-response correspondence of player i, denoted as $BR_i(s)$, is defined as $BR_i(s) = \arg\max_{s_i' \in S_i} u_i(s_i', s_{-i})$.

Definition

A strategy profile $s^* \in S$ is a Nash equilibrium (NE) $\Leftrightarrow s_i^* \in BR_i(s^*)$, $\forall\, i \in N$. Alternately, s^* is a NE $\Leftrightarrow$ for each player i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)\, \forall\, s_i \in S_i$.

Lecture Outline

Atomic and Non-atomic Congestion Games

Atomic and Non-atomic Congestion Games

Non-atomic Version

The traffic assignment problem that we discussed so far is also referred to as network game or congestion game.

Recall that we assumed that flows are infinitesimally divisible. This version is also called **non-atomic congestion game**.

Alternately, we can assume that travelers are not splittable and constrain the flows to be integers. This version is hence also called **atomic congestion game**.

Rosenthal in 1973 proved that an integer-version of the Beckmann formulation yields a pure strategy NE to the atomic congestion game.

Consider a graph $G = (V, E)$. Suppose x_{ie} is 1 if player i chooses a path with arc e and is 0 otherwise. Let for a node v, the set of outgoing and incoming arcs be denoted by $E^+(v)$ and $E^-(v)$ respectively.

Atomic and Non-atomic Congestion Games

Atomic Version

Theorem

In games derived from network equilibrium models pure-strategy NE always exist. Furthermore, any solution to the following problem is a pure-strategy NE

$$\min \sum_e \sum_{k=0}^{x_e} t_e(k)$$

$$\text{s.t.} \quad x_e = \sum_i x_{ie} \qquad \forall\, e \in E$$

$$\sum_{e \in E^+(v)} x_{ie} - \sum_{e \in E^-(v)} x_{ie} = \begin{cases} 1 & \forall\, i \in N, v = v_i^O \\ 0 & \forall\, i \in N, v \in V \backslash \{v_i^O, v_i^D\} \\ -1 & \forall\, i \in N, v = v_i^D \end{cases}$$

$$x_{ie} \in \{0,1\} \qquad \forall\, i \in N, e \in E$$

Every pure strategy NE does not necessarily solve the above optimization problem, i.e. multiple pure strategy NE may exist at least one of which may be discovered by the above formulation.

Lecture Outline

Potential Games

Potential Games

Introduction

The theory of potential games generalizes the finding that solutions to *some* games can be obtained by minimizing or maximizing a function.

Definition (Potential Game)

$\pi : A \to \mathbb{R}$ is a *potential function* and Γ is an *potential game*, if for every player i and $\forall\, a_{-i} \in A_{-i}$

$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) = \pi(a_i, a_{-i}) - \pi(a_i', a_{-i}),\ \forall\, a_i, a_i' \in A_i$$

The potential function, when maximized gives the set of pure-strategy NE ($a^* \in A$) for each player because $u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \Leftrightarrow \pi(a_i^*, a_{-i}^*) \geq \pi(a_i, a_{-i}^*)\ \forall\, a_i \in A_i, i \in N$.

Potential Games

Example

Consider the prisoner's dilemma.

	Do nothing	Detonate
Do nothing	$0, 0$	$-3, +1$
Detonate	$+1, -3$	$-2, -2$

The potential for this game may be written as $\pi = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$

For the criminals, if the civilians do nothing, $0 - 1 = -2 - (-1)$ and if they detonate, $-3 - (-2) = -1 - 0$. Check the definition for the civilians.

Potential Games

Example

The potential function for the prisoner's dilemma was $\pi = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$.
Is the potential function unique?

Not all games are potential games! How can we identify games that have a potential function?

Potential Games

Identification

Definition

The sequence $\gamma = (a^1, a^2, \ldots)$ is a *path* in A, if $\forall\, k \geq 2\, \exists$ a unique $i : a^k = (a_i, a^{k-1}_{-i})$ for some $a_i \neq a^{k-1}_i \in A_i$.

In other words, given an element of the sequence, the next strategy profile is obtained by letting a single player deviate.

Definition

For a finite path of action profiles $\gamma = (a^1, a^2, \ldots, a^K)$, let $I(\gamma, u) = \sum_{k=2}^{K} [u_{i_k}(a^k) - u_{i_k}(a^{k-1})]$, where i_k is the unique deviator at step k. We say γ is *closed* if $a^1 = a^K$. Further, if $a^l \neq a^k$ for every $l \neq k$, then γ is called a *simple closed path* (i.e., no outcome is revisited).

A path can be visualized as being traced by lattice points in a hyper-rectangle of strategy profiles allowing motion only along the axes.

The *length* of a simple closed path is the number of strategy profiles in it.

Potential Games

Identification

Theorem

The following claims are equivalent:

1. Γ *is a potential game*
2. $I(\gamma, u) = 0$ *for every finite closed path* γ
3. $I(\gamma, u) = 0$ *for every finite simple closed paths* γ *of length 4*

Note that a simple closed path of length 4 always involves only two players.

This theorem can be linked to the concept of potential used in physics. Statement 2 is similar to idea that a body does not gain/loose potential energy when taken along a closed path.

Also, uniqueness of the potential up to an additive constant is similar to the fact that potential energy of a body is a function of the reference level.

Lecture Outline

Evolution and Learning

Over the last few classes we've characterized equilibrium solutions and devised algorithms to compute them.

But do users actually converge to an equilibrium? If so, under what conditions?

This question has been well researched both in the transportation and economics communities.

Definition

A path $\gamma = (a^1, a^2, \ldots)$ is an *improvement path* if $\forall\, k \geq 2$, $u_i(a^k) > u_i(a^{k-1})$, where $i \in N$ is the deviator at step k (i.e., the deviator is required to be strictly better off).

Definition

If every improvement path generated by such myopic players is finite, then we say Γ has the *finite improvement property*(FIP).

For example, matching pennies does not have this property.

Theorem

Every finite potential game has the FIP

Proof.

For every improvement path γ, if the deviator benefits, the potential function value improves, i.e., $\pi(a^1) < \pi(a^2) < \pi(a^3) < \ldots$. As A is finite, the sequence γ has to be finite. ■

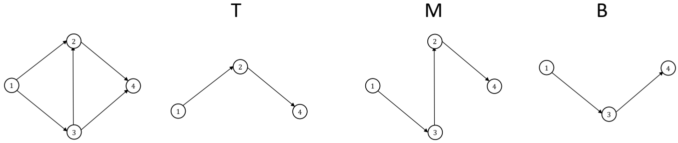
Since atomic congestion games are finite potential games, the FIP dynamic converges to an equilibrium solution.

What if all players make a move in each step?

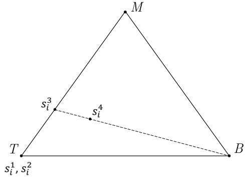
Evolution and Learning

Fictitious Play

Players can form **beliefs** about opponents (which may be treated as mixed strategies) using frequency of past actions.



Let the choices of player i on consecutive days be $T, T, M, B, \ldots$.
The beliefs of $-i$, is $(1,0,0),(1,0,0),(\frac{2}{3},\frac{1}{3},0),(\frac{1}{2},\frac{1}{4},\frac{1}{4}),\ldots$.



Evolution and Learning

Fictitious Play

At each stage players then best respond to these beliefs using a pure strategy.

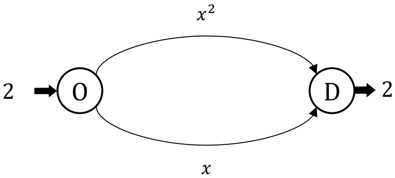
The fictitious play process converges to equilibrium, i.e., the sequence of beliefs converges to the an equilibrium solution (pure or mixed).

Theorem

Fictitious play process converges to potential maximizing solutions in finite potential games.

Evolution and Learning
Fictitious Play

Suppose two travelers are headed from O to D.



The game has two pure strategy equilibria (T, B) and (B, T), and a mixed strategy NE in which each player chooses T and B with probability $1/4$ and $3/4$ respectively.

	T	B
T	4,4	1,1
B	1,1	2,2

t	Player i $u_i(T, s_j^t)$	$u_i(B, s_j^t)$	a_i^t	$s_i^t(T)$	$s_i^t(B)$	Player j $u_j(T, s_i^t)$	$u_j(B, s_i^t)$	a_j^t	$s_j^t(T)$	$s_j^t(B)$
1	4.000	1.000	T	1.000	0.000	4.000	1.000	T	1.000	0.000
2	2.500	1.500	B	0.500	0.500	2.500	1.500	B	0.500	0.500
3	2.000	1.667	B	0.333	0.667	2.000	1.667	B	0.333	0.667
4	1.750	1.750	B	0.250	0.750	1.750	1.750	B	0.250	0.750
5	2.200	1.600	T	0.400	0.600	2.200	1.600	T	0.400	0.600
6	2.000	1.667	B	0.333	0.667	2.000	1.667	B	0.333	0.667
7	1.857	1.714	B	0.286	0.714	1.857	1.714	B	0.286	0.714
.	.	.	.	.	.	.	.	.	.	.
125	1.768	1.744	T	**0.256**	**0.744**	1.768	1.744	T	**0.256**	**0.744**

* Ties are broken in favor of the top path

t	Player i $u_i(T, s_j^t)$	$u_i(B, s_j^t)$	a_i^t	$s_i^t(T)$	$s_i^t(B)$	Player j $u_j(T, s_i^t)$	$u_j(B, s_i^t)$	a_j^t	$s_j^t(T)$	$s_j^t(B)$
1	4.000	1.000	T	1.000	0.000	4.000	1.000	T	1.000	0.000
2	2.500	1.500	B	0.500	0.500	2.500	1.500	B	0.500	0.500
3	2.000	1.667	B	0.333	0.667	2.000	1.667	B	0.333	0.667
4	1.750	1.750	B	0.250	0.750	1.750	1.750	B	0.250	0.750
5	2.200	1.600	B	0.200	0.800	1.600	1.800	T	0.400	0.600
6	2.500	1.500	B	0.167	0.833	1.500	1.833	T	0.500	0.500
7	2.714	1.429	B	0.143	0.857	1.429	1.857	T	0.571	0.429
.	.	.	.	.	.	.	.	.	.	.
125	3.928	1.024	B	**0.008**	**0.992**	1.024	1.992	T	**0.976**	**0.024**

* Ties are broken arbitrarily

Formally, $\{a^t\}_{t=1}^{\infty}$ is a fictitious play process if $\forall\, i \in N, t \geq 1, a_i^{t+1} \in BR_i(s^t)$. (best response mechanism). Let $\mathbf{1}_{\{.\}}$ represent an indicator function.

$$s_i^{t+1}(a_i) = \frac{t}{t+1} s_i^t(a_i) + \frac{1}{t+1} \mathbf{1}_{\{a_i^{t+1} = a_i\}}$$

Does this resemble anything that we have seen so far?

Historic Notes

Supplementary Reading

Fictitious play was first proposed in 1951 for zero-sum games. Most of the other results that we discussed today are due to Dov Monderer and Llyod Shapley.

Brown, G. W. (1951). Iterative solution of games by fictitious play. Activity analysis of production and allocation, 13(1), 374-376.

Monderer, D., & Shapley, L. S. (1996). Potential games. Games and economic behavior, 14(1), 124-143.

Monderer, D., & Shapley, L. S. (1996). Fictitious play property for games with identical interests. Journal of economic theory, 68(1), 258-265.

Historic Notes

Supplementary Reading

Equilibrium:
Nash (1950)
Wardrop (1952)

MSA vs Fictitious play:
Sheffi and Powell (1982)
Monderer and Shapley (1996)

Day-to-day vs Logit learning:
Cascetta (1989)
Blume (1995)

Your Moment of Zen