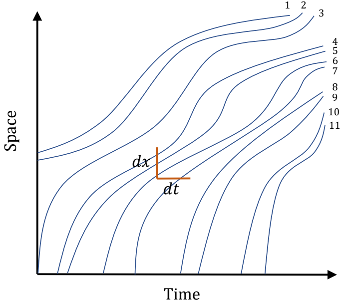
CE 269
Traffic Engineering

Lecture 8

Method of Characteristics

Previously on Traffic Engineering

Under the continuum approximation assumption, we treat $N(t, x)$ as a continuous function. Hence, we can define its partial derivatives.



$$\frac{\partial N(t,x)}{\partial x} = -k(t,x)$$
$$\frac{\partial N(t,x)}{\partial t} = q(t,x)$$

For a continuous function, we can write

$$\frac{\partial^2 N(t,x)}{\partial t \partial x} = \frac{\partial^2 N(t,x)}{\partial x \partial t}$$

Previously on Traffic Engineering

Having the fundamental diagram now gives us three sets of equations, which when solved will give the speed, density, and flow in the domain of interest.

1. $q = kv$
2. $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$
3. $q = f(k)$

Plugging the fundamental diagram equation in the conservation law, we get a PDE purely in terms of the density

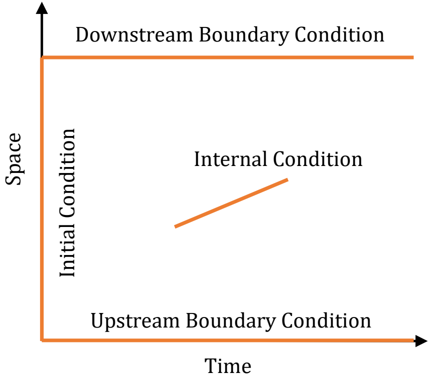
$$\frac{\partial k}{\partial t} + \frac{\partial f(k)}{\partial x} = 0$$

$$\frac{\partial k}{\partial t} + f'(k)\frac{\partial k}{\partial x} = 0$$

This equation is also called **first-order hyperbolic conservation law**.

Previously on Traffic Engineering

Solving the PDE requires some knowledge of the density function. This is prescribed in one or more of the following ways:



1. Initial Condition
2. Boundary Condition
3. Internal Condition

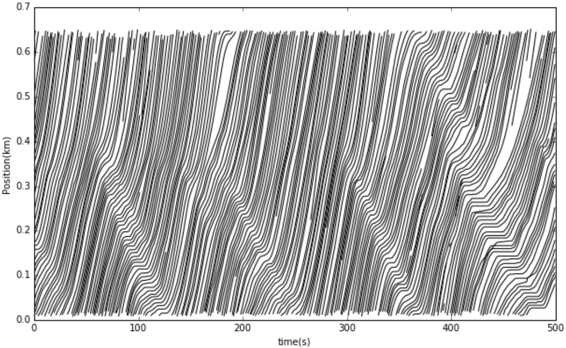
Lecture Outline

1. Waves
2. Characteristics
3. Newell's Method

Lecture Outline

Waves

Traffic on highways are known to exhibit wave-like phenomenon.



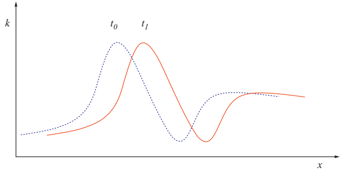
Can we find analytical solutions to the conservation equation that also capture such patterns?

Waves

Traveling Waves

Simple PDEs can be classified into different classes of equations (Transport, Laplace, Heat, Wave equations etc.)

Many equations have solutions of the form $k(t, x) = f(x - ct)$, which is called a *traveling wave*.



For positive c, $k(t, x) = f(x - ct)$ travels right over time and $k(t, x) = f(x + ct)$ travels left.

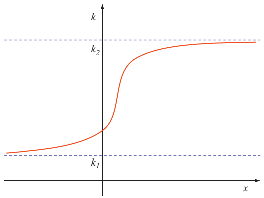
Waves

Wave fronts

A traveling wave is called a *wave front* if

$$k(t,x) = \begin{cases} k_1 & x \to -\infty \\ k_2 & x \to \infty \end{cases}$$

If $k_1 = k_2$, the solution is called a *pulse*.



Waves

General Solution

Wave equations typically have solutions which are sum of traveling waves of the following form

$$k(t,x) = F(x - ct) + G(x + ct)$$

Solve the following wave equation for $x \in (-\infty, \infty), t > 0$ with the given set of initial conditions

$$\begin{aligned} k_{tt} &= c^2 k_{xx} \\ k(0,x) &= f(x) \\ k_t(0,x) &= g(x) \end{aligned}$$

From the initial conditions on k,

$$k(0, x) = F(x) + G(x) = f(x)$$

From the initial conditions on k_t,

$$k_t(0, x) = -cF'(x) + cG'(x) = g(x)$$

Integrating from 0 to x,

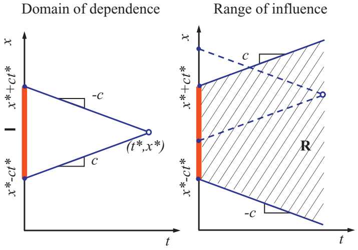
$$-F(x) + G(x) - (-F(0) + G(0)) = \frac{1}{c}\int_0^x g(y)dy$$

Solving for $F(x)$ and $G(x)$ and plugging it into the form of the general solution,

$$\begin{aligned} k(t, x) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy \end{aligned}$$

From the above equation, if we want the value of k at (t^*, x^*),

$$k(t^*, x^*) = \frac{1}{2}\Big[k(0, x^* - ct^*) + k(0, x^* + ct^*)\Big] + \frac{1}{2c}\int_{x^*-ct^*}^{x^*+ct^*} g(y)dy$$



To find k, we need two initial values and g defined on the interval $[x^* - ct^*, x^* + ct^*]$. This interval is called the *domain of dependence* of (t^*, x^*).

Likewise, we can define the *range of influence* as the points affected by the domain of dependence.

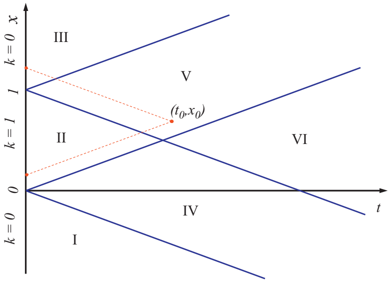
The lines with slopes c and $-c$ which help construct the domain of dependence and range of influence are called *characteristics*.

Using the characteristics, find the solution to the following wave equation for $x, t \in (0, \infty)$:

$$k_{tt} = 4k_{xx}$$
$$k(0, x) = 1 \text{ if } x \in [0, 1] \text{ and } 0 \text{ otherwise}$$
$$k_t(0, x) = 0$$

Waves

Example



Lecture Outline

Characteristics

In the conservation law $k_t + q_x = 0$, suppose $f'(k) = c$, a constant. Then, the PDE in terms of density can be written as

$$\begin{aligned} k_t + ck_x &= 0 \\ k(0,x) &= k_0(x) \end{aligned}$$

Suppose the density on the road at time $t = 0$ is given (*Cauchy problem*). As before, we are interested in finding the value of density at every (t, x).

Instead, suppose we try to estimate the density along a curve $x = x(t)$.

$$\begin{aligned} \frac{dk(t, x(t))}{dt} &= \frac{\partial k}{\partial t}\frac{dt}{dt} + \frac{\partial k}{\partial x}\frac{dx(t)}{dt} \\ &= k_t + \frac{dx(t)}{dt}k_x \end{aligned}$$

If $\frac{dx(t)}{dt} = c$, what is $\frac{dk}{dt}$?

Hence, the total time derivative of the density is constant along a curve $x = x(t)$ if $\frac{dx(t)}{dt} = c$.

In other words, the value of the density is constant along a straight line with slope c, i.e., along $x(t) = ct + x(0) = ct + x_0$.

Hence, to compute density at a point (t^*, x^*), draw the characteristic curve with slope c and look where it intersects the y axis. That is,

$$k(t^*, x^*) = k(0, x^* - ct^*) = k_0(x^* - ct^*)$$

Notice that this solution is in the form of a traveling wave. Which fundamental diagram is suited for this framework?

Characteristics

Example

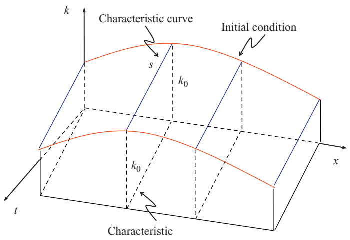
Using the method of characteristics, find the solution to the following conservation law at $(t^*, x^*) = (3, 10)$

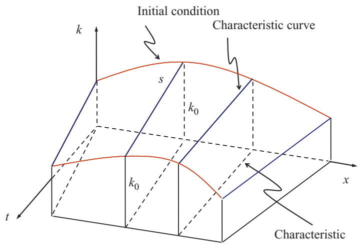
$$k_t + 2k_x = 0$$
$$k(0, x) = 2x^2 + 5$$

Characteristics

Properties

For the previous PDE, all characteristics have the same slope c and are parallel to each other. The density along these characteristics is same as the initial value.





Instead, suppose c is a function of k, i.e., we have a different fundamental diagram where $c = c(k(t,x))$.

$$\frac{dx(t)}{dt} = c(k(t,x)) \Rightarrow x = c(k_0(x_0))t + x_0$$

The characteristics in this case are straight lines but need not be parallel.

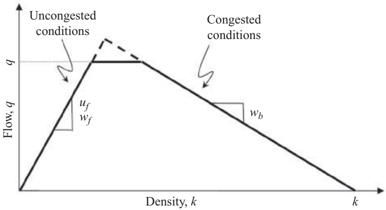
Lecture Outline

Newell's Method

Newell's Method

Introduction

The earlier analysis applies to scenarios where the slope of the flow-density relationship is constant.

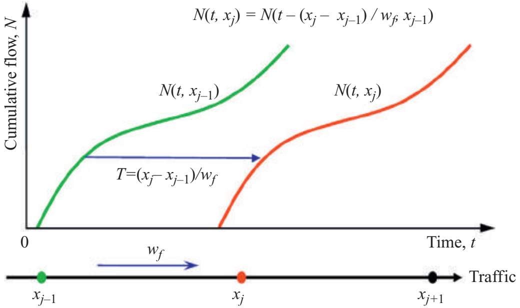


Gordon Newell extended this idea to triangular fundamental diagrams where the slopes are constant but can take two possible values depending on the density (congested or uncongested regions)

However, we do not know the traffic regime upfront to directly use the corresponding c value.

Newell's Method

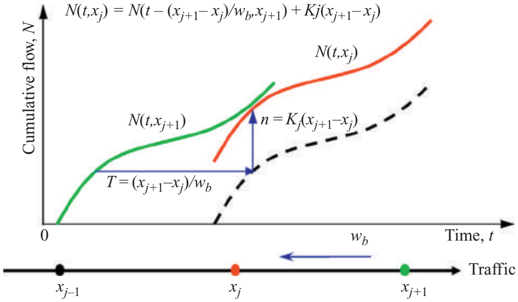
Forward Wave



$$N(t, x_j) = N(t - T, x_{j-1}) = N\left(t - \frac{x_j - x_{j-1}}{w_f}, x_{j-1}\right)$$

Newell's Method

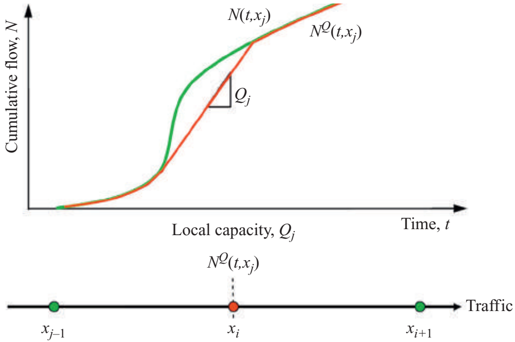
Backward Wave



$$
\begin{aligned}
N(t, x_j) &= N(t - T, x_{j+1}) + k_j(x_{j+1} - x_j) \\
&= N\left(t - \frac{x_{j+1} - x_j}{w_b}, x_{j+1}\right) + k_j(x_{j+1} - x_j)
\end{aligned}
$$

Newell's Method

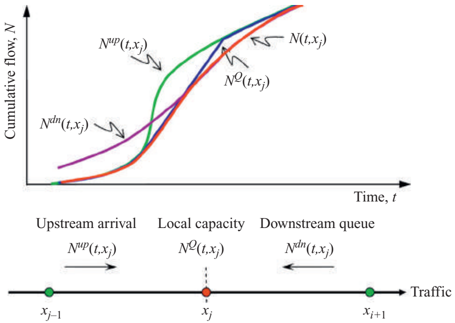
Capacity Conditions



Newell's Method

Introduction

At x_j, one cannot accommodate more vehicles than what is sent from upstream, the capacity, and what can be received downstream.

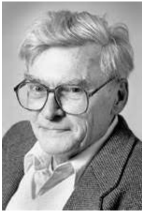


Hence, the cumulative count is the lowest of all the three conditions

$$N(t, x_j) = \min \left\{ N^{up}(t, x_k), N^Q(t, x_j), N^{dn}(t, x_j) \right\}$$

Newell's Method

Additional Reading



Newell, G. F. (1993). A simplified theory of kinematic waves in highway traffic, part I: General theory. Transportation Research Part B: Methodological, 27(4), 281-287.

Newell, G. F. (1993). A simplified theory of kinematic waves in highway traffic, part II: Queueing at freeway bottlenecks. Transportation Research Part B: Methodological, 27(4), 289-303.

https://www.jstor.org/stable/25768946

Your Moment of Zen