CE 269
Traffic Engineering

Lecture 2

Fundamental Diagrams

Previously on Traffic Engineering

Flow or Flux: Is defined as the number of vehicles passing across a point in a given amount of time T.

$$q = \frac{\Delta N}{T}$$

Headway: It is the time taken between the arrivals of the front end of successive vehicles.

$$h_i = t_i^{on} - t_{i-1}^{on}$$

Density or Concentration: It is the number of vehicles in a unit length of the road.

$$k = \frac{\Delta N}{L}$$

Spacing: It is the distance between the current vehicle and its lead vehicle.

$$s_i = x_{i-1} - x_i$$

Previously on Traffic Engineering

Speed: Speed of individual are easy to measure using mobile sensors (Why?) Using point sensors, we can compute average of speeds across multiple vehicles. This is called **time-mean speed**.

$$v_t = \frac{1}{\Delta N} \sum_i v_i$$

Speeds of individual vehicles can also be aggregated across space to derive **space-mean speed**.

$$v_s = \frac{\sum_i v_i}{L}$$

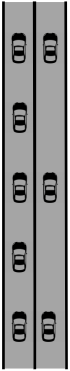
Previously on Traffic Engineering

Time- and Space-mean average speeds are usually different, especially when traffic conditions are not homogeneous.

For example, consider a two-lane highway where each car in the right lane has a speed 60 kmph and that on the left lane has a speed 30 kmph.

Suppose that the vehicles are uniformly spaced and that the flow of vehicles on both lanes is 1200 vehicles per hour.

What are the time- and space-mean average speeds?



Lecture Outline

1. Relationships between Traffic Variables
2. Single-Regime Models
3. Multi-Regime Models

Relationship between Traffic Variables

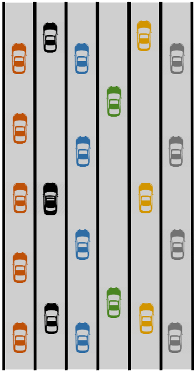
Speeds

Let us extend the earlier example to connect the time-mean and space-mean speeds.

Imagine a scenario with multiple lanes $1, \ldots, C$ each with uniform traffic with capacity q_i, density k_i, and speeds v_i.

Let $q = \sum_i q_i$ be the total flow and $k = \sum_i k_i$ be the total density.

Let $f_i = q_i/q$ and $f_i' = k_i/k$ be the proportion of observing a certain colour of vehicle across time and space.



For each lane, we can write $q_i = k_i v_i$ since the headway is q_i and spacing is v_i/q_i.

Time-mean and space-mean speeds for this setting can be written as

$$v_t = \sum_{i=1}^{C} f_i v_i$$
$$v_s = \sum_{i=1}^{C} f_i' v_i$$

Notice from the definition of the space-mean speed that

$$v_s = \sum_{i=1}^{C} \frac{k_i}{k} v_i = \frac{1}{k} \sum_{i=1}^{C} q_i = \frac{q}{k}$$

Hence, we can write $q = kv_s$ for non-homogeneous traffic but the speed v in this expression is the space-mean speed.

Relationship between Traffic Variables

Speeds

Theorem (Wardrop (1952))

Suppose σ_s^2 represents the sample variance of the space-mean speeds

$$v_t = v_s + \frac{\sigma_s^2}{v_s}$$

$$\begin{aligned} v_t = \sum_{i=1}^{C} \frac{q_i}{q} v_i = \sum_{i=1}^{C} \frac{k_i v_i^2}{q} &= \sum_{i=1}^{C} \frac{k_i}{k} \frac{v_i^2}{v_s} \\ &= \sum_{i=1}^{C} f_i' \frac{\left(v_s + (v_i - v_s)\right)^2}{v_s} \\ &= v_s + \frac{\sigma_s^2}{v_s} \text{ (Why?)} \end{aligned}$$

This result indicates that time-mean speeds are always greater than or equal to space-mean speed and both are equal when traffic is homogeneous.

Although the previous proposition connects the two speed measures, it is not that useful to estimate the space-mean speed from time-mean speed since σ_s^2 is not known. We will discuss an alternate approach soon.

The example presented applies to non-homogeneous single/multi lane traffic as well. We could imagine *subsidary streams* with different densities although it can result in shocks as we will see later in the course.

These results by Wardrop are from the same paper that discusses user equilibrium and system optimum assignment. This paper also includes methods to optimally design traffic signals.

Relationships between Traffic Variables

Point Sensor Data Revisited

Recall that the following traffic variables are best obtained from spatial sensors:

- Spacing
- Density
- Space-mean speed

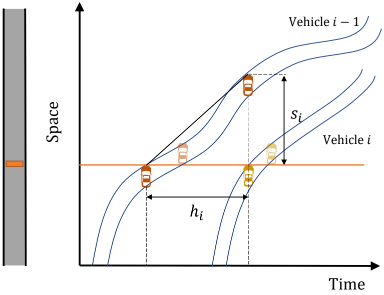
However, spatial information is often approximated from point-sensor data since it is easy to collect the latter.

Relationships between Traffic Variables

Spacing

The spacing between two vehicles can be approximated using the headway and the velocity of the lead vehicle.

$$s_i = v_{i-1} h_i$$



Note that this is an approximation since it implicitly assumes that the velocity of the lead vehicle remains the same during h_i.

Recall that density k is related to average spacing s. Hence,

$$\frac{1}{k} = s = \frac{1}{\Delta N} \sum_i v_{i-1} h_i$$

The above expression can be directly used to find the approximate density. We can also connect it with the volume in the following way

$$\frac{1}{k} = \frac{1}{\Delta N} \sum_i v_{i-1} h_i \approx \frac{1}{\Delta N} \sum_i v_i h_i$$

Suppose that **v** and **h** indices the vector of speed and headway measurements and $\widehat{Cov}$ is the sample covariance of these

$$\widehat{Cov}(\mathbf{v}, \mathbf{h}) = \frac{1}{\Delta N - 1} \sum_i (v_i - v_t)(h_i - h)$$

$$\approx \frac{1}{\Delta N} \sum_i (v_i - v_t)(h_i - h)$$

$$= \frac{1}{\Delta N} \sum_i v_i h_i - v_t h$$

Relationships between Traffic Variables

Density

Equating $\frac{1}{\Delta N}\sum_i v_i h_i$ in the above equations,

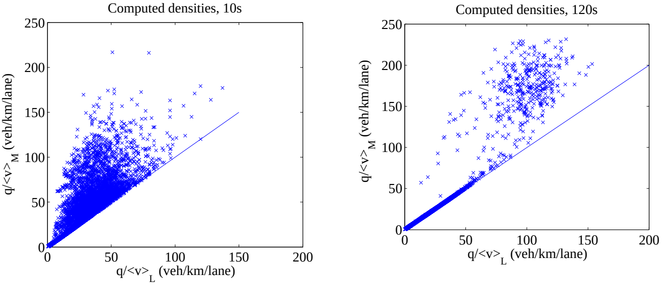
$$\begin{aligned}\frac{1}{k} &= v_t h + \widehat{cov}(\mathbf{v}, \mathbf{h}) \\ &= \frac{v_t}{q} + \widehat{Cov}(\mathbf{v}, \mathbf{h})\end{aligned}$$

Rewriting, the above expression and multiplying and dividing by $\frac{q}{v_t}$,

$$\begin{aligned}k &= \left(\frac{v_t}{q} + \widehat{Cov}(\mathbf{v}, \mathbf{h})\right)^{-1} \\ &= \frac{q}{v_t}\left(1 + \frac{q}{v_t}\widehat{Cov}(\mathbf{v}, \mathbf{h})\right)^{-1}\end{aligned}$$

In light traffic, the speeds are usually independent of headways. However, as speeds tend to zero, headways tend to be larger and the inverse portion would be > 1.

Thus, if we used v_t instead of v_s in estimating density, we would underestimate it.



Relationships between Traffic Variables

Space-mean Speed

Alternately, the space-mean speeds can be approximated using spot speeds and harmonic mean,

$$v_s \approx \frac{1}{\frac{1}{\Delta N}\sum_i \frac{1}{v_i}}$$

To derive this, we need to define a time-aggregated space-mean speed measure to use the point sensor data since it is otherwise defined at a single snapshot.

Assume that we observe the vehicles over a length L and that the speeds taken by vehicle i to traverse this portion are constant v_i

$$\begin{aligned} v_s &= \frac{\int \Delta N(t) v_s(t) dt}{\int \Delta N(t) dt} \\ &\approx \frac{\sum_i (L/v_i) v_i}{\sum_i L/v_i} \\ &= \frac{1}{\frac{1}{\Delta N}\sum_i \frac{1}{v_i}} \end{aligned}$$

The numerator and the denominator can also be interpreted as the total distance and time required for passing the segment L.

Relationships between Traffic Variables

Show that $o = lk$ in traffic where all vehicles have length l.

$$
\begin{aligned}
o &= \frac{1}{T}\sum_i (t_i^{off} - t_i^{on}) \\
&= \frac{1}{T}\sum_i l/v_i \\
&= l\frac{\Delta N}{T}\frac{\sum_i 1/v_i}{\Delta N} \\
&= l\frac{q}{v_s} = lk
\end{aligned}
$$

Exact relationships

- $q = kv_s$
- $v_t = \frac{1}{\Delta N} \sum_i v_i$
- $v_s = \frac{1}{L} \sum_i v_i$
- $v_t = v_s + \frac{\hat{\sigma_s}^2}{v_s}$, and $v_t \geq v_s$

Approximate relationships

- $s_i \approx v_{i-1} h_i$
- $k \approx \frac{q}{v_t} \left(1 + \frac{q}{v_t} \widehat{Cov}(\mathbf{v}, \mathbf{h})\right)^{-1} \approx \frac{q}{v_t}$
- $v_s \approx \frac{1}{\frac{1}{\Delta N} \sum_i \frac{1}{v_i}}$

Single-Regime Models

Single-Regime Models

Data

In addition to the relationship $q = kv$, other relationships between traffic variables were discovered from data. These are called *Fundamental Diagrams*.

They are also referred to as equilibrium models (not to be confused with user equilibrium).

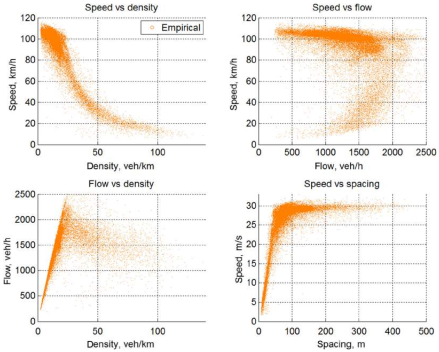
Measurements from point sensors can be used to plot the relationships between two traffic variables at a time.

Using simple curve-fitting methods, we can mathematically describe the fundamental diagram.

Single-Regime Models

Data

The following is a picture from Ni (2016) with one year of traffic data from a city in US aggregated into 5-minute intervals.

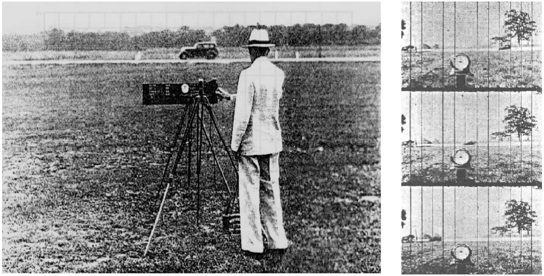


The density values are calculated from the volume and speed measurements.

Single-Regime Models

Greenshields Model

One of the earliest set of traffic measurements and models was proposed by Bruce Greenshields in 1933 using images taken from a camera.

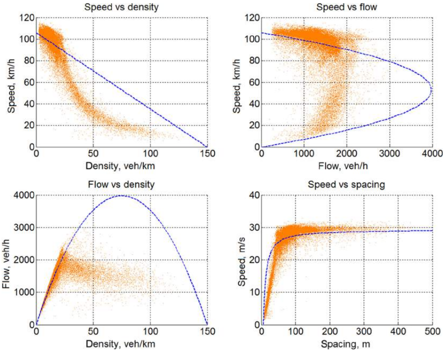


Pedersen, N. J. (2011). 75 Years of the Fundamental Diagram for Traffic Flow Theory. Transportation Research Circular No. E-C149 [PDF]

Single-Regime Models

Greenshields Model

He proposed a simple linear relationship between density k and speed v. This along with $q = kv$ gives the relationships between other pairs of variables. Why do you find so much dispersion of speeds when $k \leftarrow 0$?



Speed and Density:

$$v = v_f \left(1 - \frac{k}{k_j}\right)$$

where v_f is the free flow speed and k_j is the jam density.

Flow and Density:

$$q = v_f \left(k - \frac{k^2}{k_j}\right)$$

What is the maximum flow (capacity) according to the above equation?
$k_m = \frac{k_j}{2}$ and $q_m = \frac{v_f k_j}{4}$.

Speed and Flow:

$$q = k_j \left(v - \frac{v^2}{v_f}\right)$$

What is the speed at the maximum flow? $v_m = \frac{v_m}{2}$.

Single-Regime Models

Other Examples

In Greenshields model, v_f and k_j values can be obtained from the average length of vehicles and the speed limits. This however leads to poor fit.

Alternately, these parameters can be calibrated from the data ignoring their true meaning. Other models have also been proposed in literature to improve resemblance to real data.

Greenberg's Model

$$v = v_m \ln\left(\frac{k_j}{k}\right)$$

Where v_m is the optimal speed that can be calibrated from data. What are the disadvantages of this model?

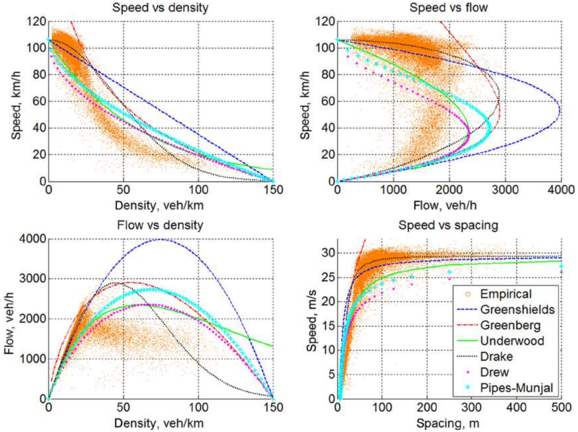
Munjal-Pipes Model

$$v = v_f\left[1 - \left(\frac{k}{k_j}\right)^n\right]$$

For $n = 1$, this resembles the Greenshields model.

Single-Regime Models

Summary



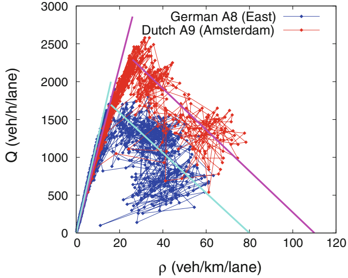
Lecture Outline

Multi-Regime Models

Multi-Regime Models

Drawbacks of Single-Regime Models

The flow-density curves often tend to exhibit different behaviour in the un-congested and congested portions.



Phenomena such as *capacity drop* and *dispersion* are commonly observed. This motivates the need for using more parameters or different functions for different regimes of the fundamental diagram.

Multi-Regime Models

Examples

Some of these models are connected with microscopic traffic models that we will see later in the course.

Newell's Model:

$$v = v_f \left(1 - \exp\left(\frac{\lambda}{v_f}\left(\frac{1}{k} - \frac{1}{k_j}\right)\right)\right)$$

The parameter λ is related to the speed-spacing curve.

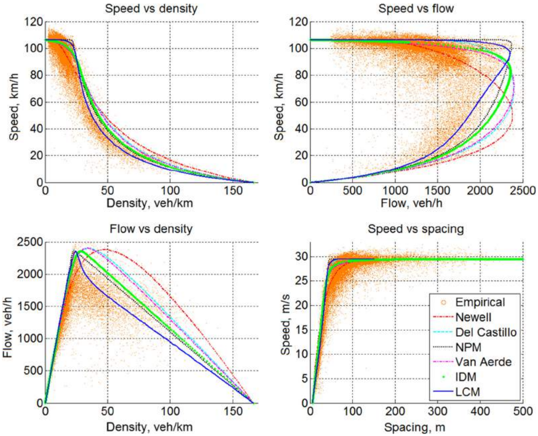
Intelligent Driver Behaviour Model:

$$k = \frac{1}{(s_0 + vT)\left[1 - \left(\frac{v}{v_f}\right)^{\delta}\right]^{-1/2}}$$

where s_0 represents the jam distance, T is the safe time headway, and δ denotes the acceleration exponent.

Multi-Regime Models

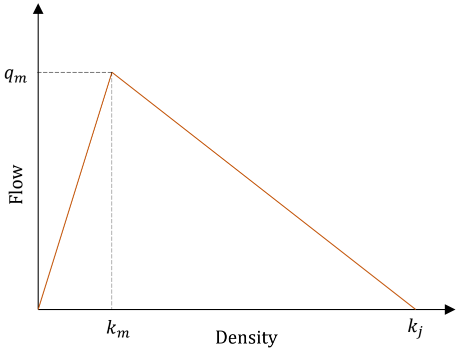
Summary



Multi-Regime Models

Exercise

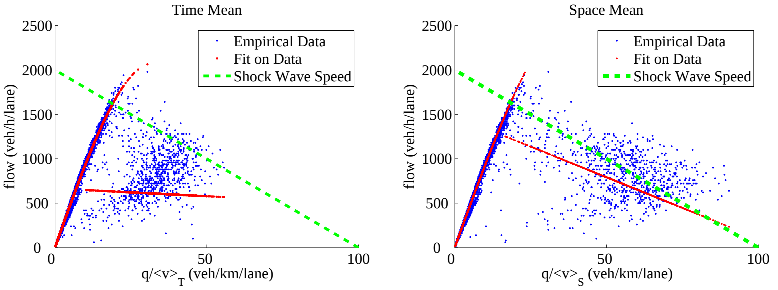
Plot the other fundamental diagrams associated with the following triangular flow-density curve.



Multi-Regime Models

Caution

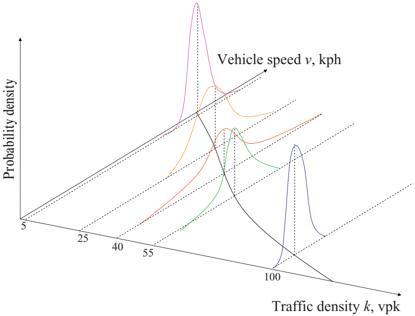
Using space vs. time-mean speeds can have an impact on the flow-density plots and also the calibrated fundamental diagrams.



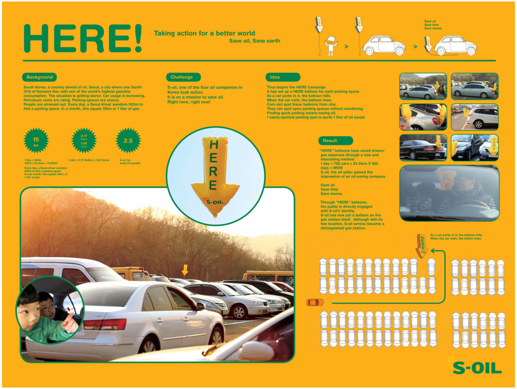
Multi-Regime Models

Stochastic FD

Other extensions to stochastic fundamental diagrams also help solve issues associated with dispersion caused due to heterogeneity in driver behaviour.



Your Moment of Zen



https://youtu.be/ccM8RLLKasw