CE 269
Traffic Engineering

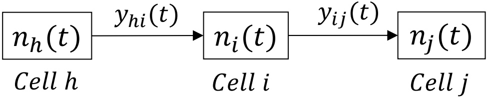
Lecture 19

Dynamic Traffic Assignment

Previously on Traffic Engineering

The variables used in CTM are:

- $y_{ij}(t)$: Denotes the flow from cell i to cell j in $[t, t + \Delta t] \equiv [t, t + 1]$.
- $n_i(t)$: Number of vehicles in cell i at time t
- N_i: Maximum number of vehicles that can fit in cell i.



Conservation of flow requires

$$n_i(t + 1) = n_i(t) + y_{hi}(t) - y_{ij}(t)$$

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Remember that these iterates give us the flow between cells on a link.

$$A \longrightarrow B$$

$$\boxed{n_1(t)} \xrightarrow{y_{12}(t)} \boxed{n_2(t)} \xrightarrow{y_{23}(t)} \boxed{n_3(t)} \xrightarrow{y_{34}(t)} \boxed{n_4(t)} \xrightarrow{y_{45}(t)} \boxed{n_5(t)}$$

In fact, one can think of cells as miniature links in series and notice that the sending and receiving flow expressions are captured in what we derived.

$$y_{ij}(t) = \min\left\{ n_i(t), q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f} \right\}$$

$$y_{ij}(t) = \min\left\{ \min\left\{ n_i(t), q_{max}\Delta t \right\}, \min\left\{ q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f} \right\} \right\}$$

The first minimum is the sending flow of cell i and the second minimum is the receiving flow of cell j.

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The routing decisions are captured using proportions just as done in the diverge case.

At each node, we keep track of an *allocation* or *routing* matrix $P_i(t)$ of size $|A_i| \times |A_i^{-1}|$ which specifies what fraction of travelers on an outgoing link come from different incoming links.

Let A_i and A_i^{-1} be the adjacency and the inverse adjacency list. The elements of the matrix are $p_{ij,hi}(t)$ and satisfy $\sum_j p_{ij,hi}(t) = 1 \ \forall \ h \in A_i^{-1}$

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Suppose the sending flows on the incoming links be denoted by $S_{hi}(t)$ and receiving flows by $R_{ij}(t)$.

Let the actual flows on the incoming and outgoing links be represented by y_{hi} and y_{ij}, respectively.

Then, the following constraints must hold.

$$y_{hi}(t) \leq S_{hi}(t) \ \forall \ h \in A_i^{-1}$$

$$y_{ij}(t) \leq R_{ij}(t) \ \forall \ j \in A_i$$

How are the y variables related to the allocation matrix?

$$y_{ij}(t) = \sum_{h \in A_i^{-1}} p_{ij,hi}(t) y_{hi}(t)$$

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Using this optimization problem, derive the results that we saw earlier for a pair of links in series and a simple merge and a diverge.

$$
\begin{aligned}
& \max \sum_{j \in A_i} y_{ij}(t) \\
\text{s.t.} \quad & y_{ij}(t) = \sum_{h \in A_i^{-1}} p_{ij,hi}(t) y_{hi}(t) && \forall\, (i,j) \in A \\
& y_{hi}(t) \leq S_{hi}(t) && \forall\, h \in A_i^{-1} \\
& y_{ij}(t) \leq R_{ij}(t) && \forall\, j \in A_i \\
& y_{ij}(t) \geq 0 && \forall\, (i,j) \in A
\end{aligned}
$$

Does this model handle scenarios where the sending and receiving flows are zero in a merge and diverge?

CTM has also been extended to signalized intersection with priority rules.

Lecture Outline

1. Dynamic User Equilibrium
2. Time-Dependent Shortest Paths
3. Shifting Flows

Lecture Outline

Dynamic User Equilibrium

Dynamic Traffic Assignment: In the last semester, we studied static traffic assignment models in detail. They are mathematically attractive but they do not represent time dynamics.

- In reality, users depart at different times.
- Traffic builds and dissipates on a link in a more complex fashion and can create shockwaves. None of this is modeled in static TAPs.

Dynamic traffic assignment is an extension of the static approaches in which macroscopic traffic flow models are used instead of link delay functions.

Dynamic User Equilibrium

Definition

The Wardrop principle in this setting states that '**all users departing at the same time have equal and minimal travel times**'.

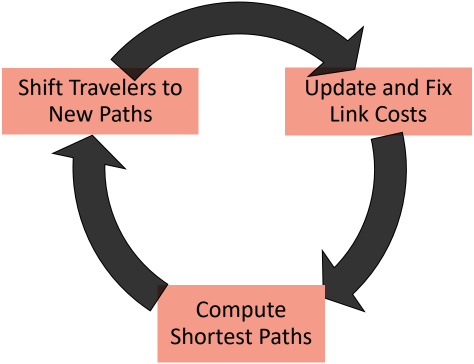
One can also formulate models in which users are allowed to switch their departure times.

However, existence and uniqueness of DTA is extremely difficult to establish because of non-linearity and discontinuities of the delay functions. DTA is built using a simulation framework and empirically most network converge.

Dynamic User Equilibrium

Overview

The equilibrium procedure is similar but modifications are required at each step.



- **Update and Fix Link Costs:** This is carried out using simulators.
- **Compute Shortest Paths:** Since link delays vary across time, we now have to calculate time-dependent shortest paths.
- **Shift Travelers:** This step is typically done using MSA or gradient projection-like methods. Computing travel time derivatives is challenging.

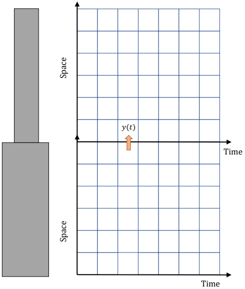
The microscopic and macroscopic models that we saw so far allow us to compute the travel times on the links in the network **over time**.

Given the departure times of the travelers in the network and their path choices, the goal is to find the travel times that they **experience** (as opposed to the **instantaneous** time).

Note that the dynamic traffic assignment models are more demanding. You need to know the demand distribution over time and more details on the road geometry (backward wave speeds, capacity, etc.)

Time-Dependent Shortest Paths

The traffic flow models that we saw so far tell us the travel time on each link for different departure times at the tail node given a fixed set of path flows.



We'll now find the time-dependent shortest paths and shift travelers from longer paths to shorter ones **for each departure time step**.

It is useful to distinguish networks as first-in-and-first-out (FIFO) or non-FIFO as the algorithms for finding shortest paths are slightly different.

FIFO networks satisfy the property if $t < t'$ then $t + c_{ij}(t) < t' + c_{ij}(t')$. That is if A enters a link before B, A will arrive at j before B.

An important aspect to consider in time-dependent graphs is waiting.

Some graphs allow travelers to wait at intermediate nodes and waiting may also be optimal. Can you think of an example?

In FIFO graphs, waiting will only increase the travel time. Hence, one can ignore it and a Dijkstra's-like algorithm can be used.

Suppose we depart from an origin r at t_0

Algorithm 1 MODIFIED DIJKSTRA'S ALGORITHM(G, r, t_0)

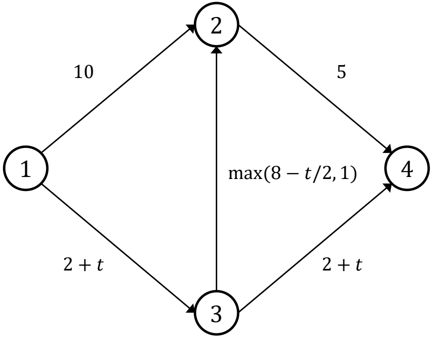
Step 1: Initialize
$S = \emptyset$, $\bar{S} = N$
$\mu_r = t_0, \pi_r = r$
$\mu_i = \infty, \pi_i = -1 \, \forall \, i \in N \backslash \{r\}$

Step 2:
while $\bar{S} \neq \emptyset$ **do**
  $i = \arg\min_{j \in \bar{S}} \mu_j$
  $S = S \cup \{i\}$, $\bar{S} = \bar{S} \backslash \{i\}$
  for $j : (i, j) \in A$ **do**
    if $\mu_j > \mu_i + c_{ij}(\mu_i)$ **then**
      $\mu_j = \mu_i + c_{ij}(\mu_i)$
      $\pi_j = i$
    end if
  end for
end while

Time-Dependent Shortest Paths

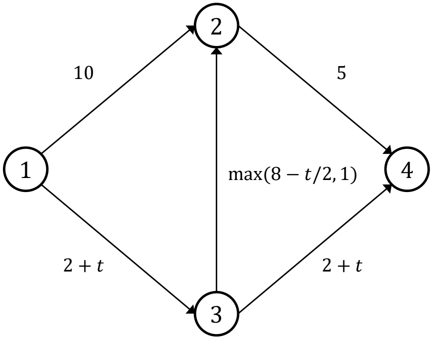
Example

Is the following network a FIFO graph?

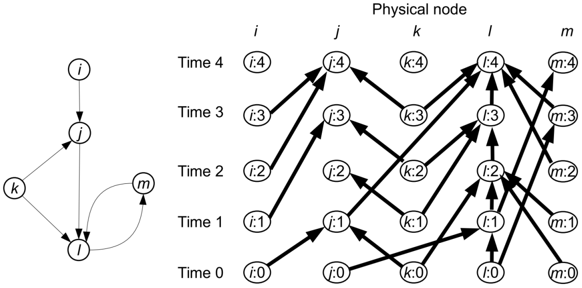


Find the shortest paths to the other nodes if we depart from node 1 at $t_0 = 4$.

What is the instantaneous shortest path in this network if we depart from node 1 at $t_0 = 4$?



In general networks, we can construct a time expanded version of the graph and solve for the optimum labels with minimal modifications to the algorithms we saw earlier.



We maintain a copies of each node for every time step and connect arcs between nodes to reflect the time-varying travel times. Is the above figure a FIFO graph?

Time-Dependent Shortest Paths

General TDSP

Instead of a single node label μ_i, we not have a node label for every departure time μ_i^t. Equivalently, we may assume that we have a node label for every node in the time-expanded graph.

Solving a regular shortest path problem in the time-expanded graph will give us the TDSPs. Should we maintain a SEL or can we do better?

Time-expanded graphs are acyclic and hence we scan nodes in increasing order of time.

Algorithm 2 TDSP(G)

Step 1: Initialize

$\mu_r^{t_0} = 0, \pi_r^{t_0} = r$

$\mu_i^t = \infty, \pi_i^t = -1 \, \forall \, i \in N \backslash \{r\}, t \neq t_0$

Step 2:

for $t \in \{1, 2, \ldots, T\}$ **do**

    for Arcs from (i, t) to (j, t') **do**

        if $\mu_j^{t'} > \mu_i^t + c_{ij}(t)$ **then**

            $\mu_j^{t'} = \mu_i^t + c_{ij}(t)$

            $\pi_j^{t'} = i$

        end if

    end for

end for

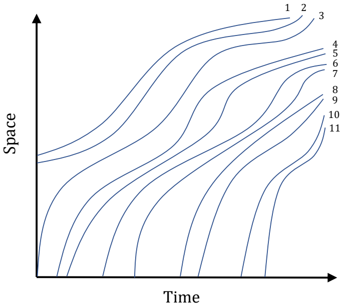
Lecture Outline

Shifting Flows

Shifting Flows

Updating Link Travel Times

Recall the space-time plots that we constructed to motivate the traffic flow models.



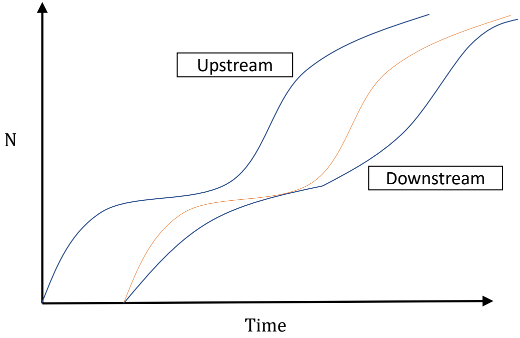
We can redraw this in a different manner only considering cumulative counts at the upstream and downstream ends of the link.

Here, we assume that the time-dependent demand is known and we carry out a network loading procedure using CTM or LTM.

Shifting Flows

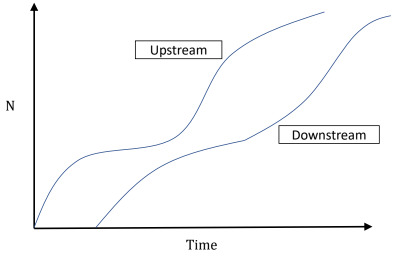
Updating Link Travel Times

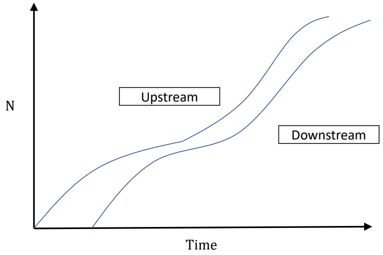
These are the plots of the cumulative count functions at the upstream and downstream ends of a link $N(0, t)$ and $N(L, t)$.



How do you find the travel time on the link for different departure times?

We can compute the time-dependent path travel times by piecing link travel times.





Shifting Flows

Method of Successive Averages

After time-dependent travel times are computed for a given path flow solution, we shift flow from longer paths to shorter ones just like in static traffic assignment.

We keep track of a path flow matrix for different departure times Y. Imagine a network with a single OD pair.

$$Y = \begin{bmatrix} y_1(1) & y_2(1) & \dots & y_P(1) \\ y_1(2) & y_2(2) & \dots & y_P(2) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(T) & y_2(T) & \dots & y_P(T) \end{bmatrix}$$

The rows represent departure times and the columns indicate different paths between the OD pair.

From CTM and TDSP, we also know the travel time (and the shortest) on the paths for different departure times.

$$Y = \begin{bmatrix} \tau_1(1) & \tau_2(1) & \dots & \tau_P(1) \\ \tau_1(2) & \tau_2(2) & \dots & \tau_P(2) \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1(T) & \tau_2(T) & \dots & \tau_P(T) \end{bmatrix}$$

We then find the all-or-nothing assignment by loading all travelers on the time-dependent shortest path $\hat{Y}$. Finally, the path flows are iteratively updated using

$$Y^{k+1} = \eta_k \hat{Y}^k + (1 - \eta_k) Y^k$$

The step size is typically set to $1/k$ where k is the iteration number.

Shifting Flows

Method of Successive Averages

One can mathematically prove that MSA converges to the equilibrium solutions in the case of static traffic assignment.

However, in DTA we do not have an equivalent convex optimization problem to make use of and hence convergence is not theoretically guaranteed but empirically most networks exhibit convergence (sometimes after days).

Convergence can be measured using measures similar to relative gap and average excess cost.

Recall the path flow update mechanism for gradient projection

$$\Delta Y(t) = \min \left\{ Y_p(t), \frac{\tau_p(t) - \tau_{p^*}(t)}{\sum_{(i,j) \in \hat{A}} t'_{ij}} \right\}$$

In DTA, we perform similar updates for every departure time step. The path travel times are computed as shown earlier, but we still need the travel time derivatives.

Shifting Flows

Gradient Projection

Travel time derivatives on links depend on where we are in the fundamental diagram. If we are in the free-flow region, adding an extra vehicle will not increase travel time.

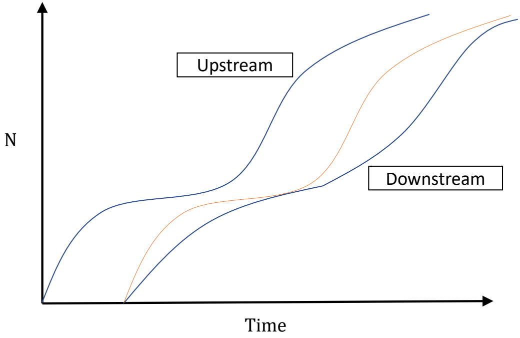
But if we are in the congested regime, adding an extra vehicle will increase the travel time. The question is by how much? $1/q$ where q is the flow or if you are using CTM, we use $1/y$ variables.

One needs to be careful of the time steps at which the derivatives are computed.

MSA and GP

Gradient Projection

Can we identify if there is a queue or not using the cumulative counts figure?



After computing derivatives, we shift flows and repeat the process of running CTM, recalculate time-dependent shortest paths, flow shifts and repeat.

Shifting Flows

A Parting Note

DTA does model travel demand, traffic congestion, and route choices at a very fine scale. Does this imply that it is a better model?

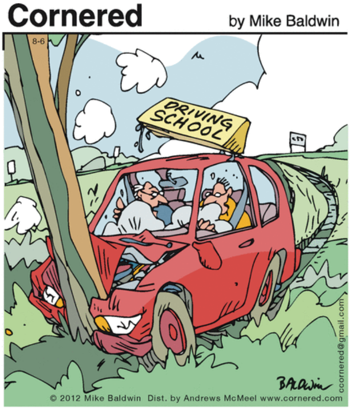
Additional Reading

Boyles, S. Lownes, N. E., & Unnikrishnan, A. (2020). Transportation Network Analysis. (Part III)

Chiu, Y. C., Bottom, J., Mahut, M., Paz, A., Balakrishna, R., Waller, S., & Hicks, J. (2011). Dynamic traffic assignment: A primer (Transportation Research Circular E-C153).

Peeta, S., & Ziliaskopoulos, A. K. (2001). Foundations of dynamic traffic assignment: The past, the present and the future. Networks and spatial economics, 1(3), 233-265.

Your Moment of Zen



"There isn't a 'Play Again' button."