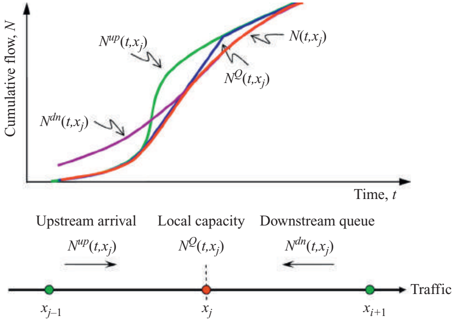
CE 269
Traffic Engineering

Lecture 10

Cell Transmission Model

Previously on Traffic Engineering

At x_j, one cannot accommodate more vehicles than what is sent from upstream, the capacity, and what can be received downstream.



Hence, the cumulative count is the lowest of all the three conditions

$$N(t, x_j) = \min\left\{N^{up}(t, x_k), N^Q(t, x_j), N^{dn}(t, x_j)\right\}$$

Previously on Traffic Engineering

For the case of the triangular fundamental diagram, we have only two characteristics to deal with and the correct cumulative count is obtained from the most restrictive initial/boundary conditions.

Case I: If $f' = w_f$, $q = kw_f$ and hence

$$N(t, x) = N(t_U, x_U)$$

That is, we trace the same vehicle as we move along the characteristics.

Case II: If $f' = w_b$, $k + q/w_b = k_j$, and hence

$$\begin{aligned} N(t, x) &= N(t_C, x_C) + (q + kw_b)(t - t_C) \\ &= N(t_C, x_C) + k_j w(t - t_C) \\ &= N(t_C, x_C) + k_j(x_C - x) \end{aligned}$$

In this case, the cumulative count increases at the rate of the jam density.

Previously on Traffic Engineering

Thus, for any (t, x),

$$N(t, x) = \min \left\{ N(t_U, x_U), N(t_C, x_C) + k_j(x_C - x) \right\}$$

Instead of tracking the cumulative counts, we could draw characteristics and work with densities.

But as we will see shortly, shock waves can make this procedure difficult. This approach on the other hand can be applied oblivious to the existence of shock waves.

Lecture Outline

1. Numerical Methods
2. Cell Transmission Model
3. Example

Lecture Outline

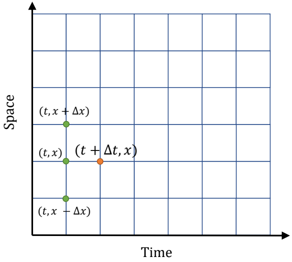
Numerical Methods

Numerical Methods

Introduction

The LWR model PDE $\frac{\partial k}{\partial t} + \frac{\partial f(k)}{\partial x} = 0$ can be approximated using Lax Friedrich-type finite difference method in the following way

$$\frac{k(t+\Delta t, x) - k(t, x)}{\Delta t} + \frac{f(k(t, x+\Delta x)) - f(k(t, x-\Delta x))}{2\Delta x} = 0$$



There are other efficient ways to approximate the PDE, which will be discussed now.

Numerical Methods

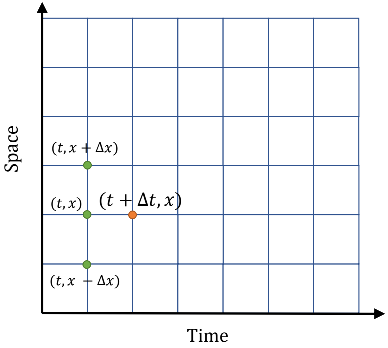
Going Forward

So far, we have just understood what happens to traffic on a single link.

Given, ICs or BCs, one can predict the density, volume, and speed (which can then be processed to find the travel times on the link if we enter it at different times).

We will see how solutions from one link can be used as boundary conditions to others and we can scale this method at a network level. (Imagine space-time plots for each link which are all linked together.) This step is referred to as **network loading**.

The model that we saw in the last lecture gives us the density at different points in space and time (which can also be used to estimate the the cumulative count values).

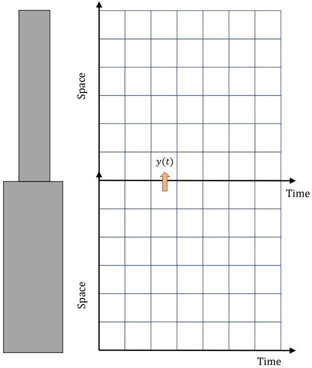


The cumulative counts determine the flow into or out of the link in each time interval, i.e., $y(t) = N(t, x) - N(t + \Delta t, x)$ as shown.

Numerical Methods

Extending LWR to a Network

It is important to note that the flow between boundaries are interlinked due to the network structure. For example, consider the following instance.

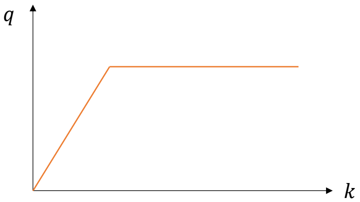


In general, both conditions downstream and upstream can affect traffic at a location.

The **sending flow** or **demand** out of a link in time interval $[t, t + \Delta t]$ denoted as $S(t)$ is defined as the flow that can leave the link if the downstream end is connected to a reservoir of infinite capacity.

What factors influence sending flows?

- Capacity of the link
- Density of vehicles currently on the link (we may have an uncongested scenario)



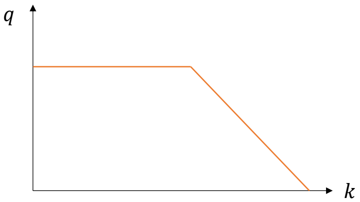
Numerical Methods

Sending and Receiving Flows

Likewise, the **receiving flow** or **suppy** into of a link in time interval $[t, t + \Delta t]$ denoted as $R(t)$ is defined as the flow that can enter the link if the upstream end is connected to a reservoir of infinite capacity.

What factors influence receiving flows?

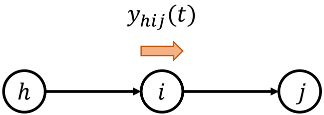
- Capacity of the link
- Density of vehicles currently on the link (we may have an congested scenario which prohibits vehicles from entering)



Numerical Methods

Links in Series

Consider the simplest scenarios when we have two links in series.



The flow that goes from (h, i) to (i, j) is

$$y_{hij}(t) = \min \left\{ S_{hi}(t), R_{ij}(t) \right\}$$

We will ignore t to keep the notation simple, but remember that all these terms vary over time.

Lecture Outline

Cell Transmission Model

Cell Transmission Model

Finite-Difference Approximation

The road space is assumed to be divided into smaller cells and we track the density in each cell and the flows between adjacent cells. (Godunov Scheme)

This method is also called the Cell Transmission Model (CTM). There are other similar models such as the Link Transmission Model (LTM).

The cells in CTM must be carefully chosen. For a given time step Δt, we set the size of the cell to be the equal to the distance a vehicle will travel in free flow conditions, i.e., $\Delta x = v_f \Delta t$. This assumption has a couple of advantages:

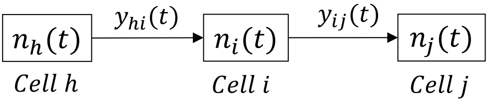
- Vehicles cannot skip cells in one time step. They can only move from one cell to the next in one time step.
- A condition called Courant-Friedrich-Lewy (CFL) is satisfied which guarantees that this method is a stable solution to the PDE.

Cell Transmission Model

Notation

The variables used in CTM are:

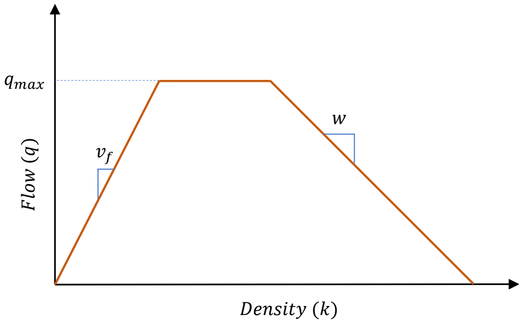
- $y_{ij}(t)$: Denotes the flow from cell i to cell j in $[t, t+\Delta t] \equiv [t, t+1]$.
- $n_i(t)$: Number of vehicles in cell i at time t
- N_i: Maximum number of vehicles that can fit in cell i.



Conservation of flow requires

$$n_i(t+1) = n_i(t) + y_{hi}(t) - y_{ij}(t)$$

A trapeziodal fundamental diagram is usually used in CTM



v_f is the free-flow speed, w is the backward wave velocity, and q_{max} is the capacity of the link.

Just as the triangular fundamental diagram, this can be mathematically represented as

$$q = \min\left\{v_f k, q_{max}, (k_j - k)w\right\}$$

Cell Transmission Model

Fundamental Diagram

$$q = \min\left\{v_f k, q_{max}, (k_j - k)w\right\}$$

Multiply both sides of the equation with Δt

$$q\Delta t = \min\left\{v_f k\Delta t, q_{max}\Delta t, (k_j\Delta t - k\Delta t)w\right\}$$

The equivalent formula in terms of y and n variables is given by

$$y_{ij}(t) = \min\left\{n_i(t), q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f}\right\}$$

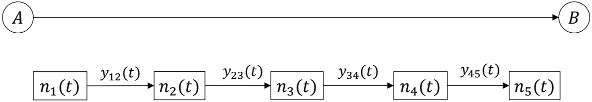
Can you explain this expression in words? The number of vehicles which move from cell i to cell j is limited by the

- Existing number of vehicles $n_i(t)$
- Flow at capacity $q_{max}\Delta t$
- Available space in the downstream cell j. Remember this is congested, so the velocity is lower and hence the factor w/v_f.

Cell Transmission Model

Sending and Receiving Flows

Remember that these iterates give us the flow between cells on a link.



In fact, one can think of cells as miniature links in series and notice that the sending and receiving flow expressions are captured in what we derived.

$$y_{ij}(t) = \min\left\{ n_i(t), q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f} \right\}$$

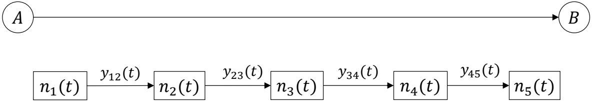
$$y_{ij}(t) = \min\left\{ \min\left\{ n_i(t), q_{max}\Delta t \right\}, \min\left\{ q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f} \right\} \right\}$$

The first minimum is the sending flow of cell i and the second minimum is the receiving flow of cell j.

Cell Transmission Model

Sending and Receiving Flows

The sending and receiving flows are calculated for the last and the first cells of the link



For instance, the sending flow of link (A, B) is calculated from cell 5.

$$S_{AB}(t) = \min \left\{ n_5(t), q_{max} \Delta t \right\}$$

$$R_{AB}(t) = \min \left\{ q_{max} \Delta t, \left(N_1 - n_1(t) \right) \frac{w}{v_f} \right\}$$

Using the S and R expressions and the network topology, we update the flows across cells in the next link.

All cell occupancies n and flows y at time step t are updated with values from time step $t - 1$ and so on.

Lecture Outline

Example

Example

Three Cell Link

Consider a link with three cells. Suppose that time is divided into 1 second intervals.

- At most 10 vehicles can move from one cell to the next cell in one time step, i.e., $q_{max}\Delta t = 10$.
- The maximum number of vehicles that can fit a cell $N_i = 30$.
- $w/v_f = 2/3$
- The demand for vehicles trying to enter the link is known $d(t)$.

Now also assume that at the downstream end, a traffic light is red from $t = 0$ to $t = 9$ and will turn green forever at $t = 10$. Use the spreadsheet provided to calculate cell occupancies over time.

Additional Reading

Daganzo, C. F. (1994). The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. Transportation Research Part B: Methodological, 28(4), 269-287.

Daganzo, C. F. (1995). The cell transmission model, part II: network traffic. Transportation Research Part B: Methodological, 29(2), 79-93.

Your Moment of Zen

From Daganzo's 1984 paper on 'The length of tours in zones of different shapes':

Acknowledgement—Gordon F. Newell shared his ideas with me, and some of the errors in this paper are probably his.