# CE 211
# Mathematics for Engineers

Lecture 5

**Continuous Random Variables**

# Previously on Mathematics for Engineers

Note the probability measure is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ where you can think of $\mathcal{F}$ as $2^\Omega$, whereas the random variable $X$ is another function $X : \Omega \to \mathbb{R}$.

As seen in the previous examples, for subsets $S \subset \mathbb{R}$, we can find an event $A \in \mathcal{F}$ such that $X^{-1}(S) = A = \{\omega \in \Omega | X(\omega) \in S\}$.

Hence, the following probabilities are the same

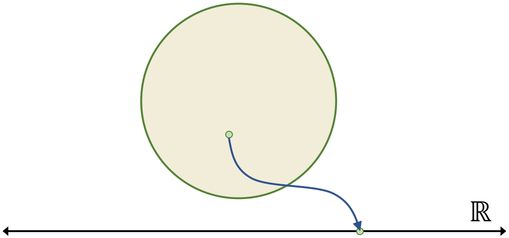
$$\mathbb{P}(X \in S) = \mathbb{P}(X^{-1}(S)) = \mathbb{P}(A) = \mathbb{P}(\{\omega \in \Omega | X(\omega) \in S\})$$

Be careful to not write $X(A)$ and $\mathbb{P}(S)$, where $A \in \mathcal{F}$ and $S \subset \mathbb{R}$ (unless of course $\Omega = \mathbb{R}$).

# Previously on Mathematics for Engineers

Continuous random variables are ones which are defined on uncountable sample spaces.

For example, $X$ could represent the location of a randomly thrown dart on the interval $[0, 1]$ in which case it can be written as $X : [0, 1] \to \mathbb{R}$ or on a two-dimensional circle of some radius, i.e., $X : C \to \mathbb{R}$, where $C = \{(x, y) | x^2 + y^2 \leq r\}$ etc.



# Previously on Mathematics for Engineers

Just like the discrete case, we define a probability density function but with a small twist since the probability of observing a singleton event is 0.

**Definition (Probability Density Function)**

The probability density function (PDF) of a continuous random variable is denoted as $f_X(x)$ and is defined as

$$f_X(x)dx = \mathbb{P}\big(X \in [x, x + dx]\big)$$

Thus, the probability that the random variable lies in a subset $S$ is given by

$$\mathbb{P}(X \in S) = \int_{x \in S} f_X(x)dx$$

Since the probability that $X$ equals any value is 0, the above definition could have been written using $(x, x + dx]$, $[x, x + dx)$, or $(x, x + dx)$.

# Previously on Mathematics for Engineers

The definition of CDF for continuous random variables remains unchanged

**Definition (Cumulative Distribution Function)**

The cumulative distribution function (CDF) of a random variable $X$ is denoted by $F_X(x)$ and is defined as

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(x)dx$$

# Lecture Outline

1. The Problem
2. Expectation and Variance
3. Uniform Distribution
4. Exponential Distribution
5. Normal Distribution
6. Other Distributions
7. A Solution

# Lecture Outline

## The Problem

Suppose you are to choose between the following two options:

1. I toss a biased coin and with probability 0.8 I'll give you ₹4000 and with probability 0.2 I'll give you nothing.
2. I'll give you ₹3000 with probability one.

Which one would you choose?

# The Problem

## Prospect Theory

Now choose between these options:

1. I toss a biased coin and with probability 0.8 I'll take ₹4000 from you and with probability 0.2 I'll take nothing.
2. I'll take ₹3000 from you with probability one.

Which one would you choose? Why did some of you change your mind?

# Lecture Outline

## Expectation and Variance

# Expectation and Variance

## Expectation

The notion of expectation and variance can be extended to continuous random variables by replacing summations with integrals

**Definition (Expectation)**

The expected value of a random variable $X$ is denoted by $\mathbb{E}(X)$ or $\mu_X$ and is defined as

$$\mathbb{E}(X) = \int_{x \in R_X} x f_X(x) dx$$

Note that by definition of CDF,

$$\frac{dF_X(x)}{dx} = f_X(x)$$

.

Hence, the formula for expectation can also be written as

$$\mathbb{E}(X) = \int_{x \in R_X} x dF_X(x)$$

Just like the discrete case, we can also define expectation of functions of random variables $f(X)$

**Definition (Expectation of Functions)**

The expectation of a function of random variable $g(X)$ is denoted by $\mathbb{E}(g(X))$ or $\mu_{g(X)}$ and is defined as

$$\mathbb{E}(g(X)) = \int_{x \in R_X} g(x) f_X(x) dx$$

The extend of dispersion or spread around the mean is captured by variance.

**Definition (Variance)**

The variance of a random variable $X$ is denoted by $V(X)$, $\mathrm{Var}(X)$, or $\sigma_X^2$ and is defined as

$$V(X) = \mathbb{E}\left((X - \mu_X)^2\right) = \int_{x \in R_X} (x - \mu_X)^2 f_X(x) dx$$

The term $\sigma_X = \sqrt{V(X)}$ is also called the standard deviation of $X$.

# Expectation and Variance

## Functions of Random Variables

The following results also hold for the continuous case

**Claim**

- *If a and b are constants,* $\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$
- $V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
- $V(g(X)) = \mathbb{E}\Big((g(X) - \mu_{g(X)})^2\Big)$
- *If a and b are constants* $V(aX + b) = a^2 V(X)$

# Uniform Distribution

# Uniform Distribution

## Motivating Example

The uniformly distributed random variable on an interval is one of the simplest continuous random variables.

The following situations are examples which can be modelled using such random variables.

- A dart is thrown at random on a line. The position at which it lands can be modelled using a uniform distribution.
- A bus route operates at a certain frequency, say 15 min. The time at which a passenger arrives between two consecutive bus arrivals can be assumed to be uniformly distributed.

# Uniform Distribution

## Probability Density Function

**Definition**

Suppose $X \sim U(a, b)$, then the PDF of $X$ is defined as

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- What is the support of this random variable?
- Is this a valid PDF? That is, is the area under this curve 1?
- What is its cumulative distribution function?

# Uniform Distribution

## Cumulative Density Function

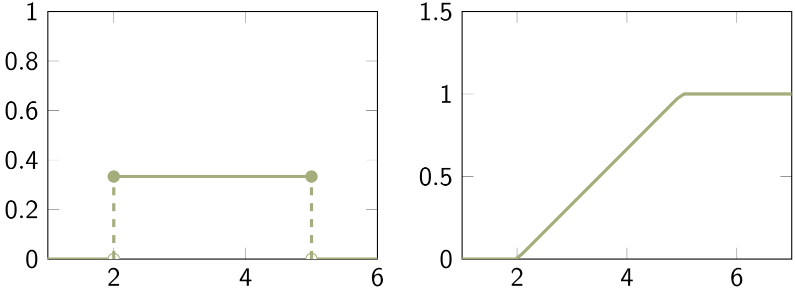
**Claim**

*Suppose $X \sim U(a, b)$, then the CDF of $X$ is*

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{otherwise} \end{cases}$$

Check if $dF_X(x)/dx = f_X(x)$.

The PDF and CDF of a uniformly distributed random variable on [2,5] is shown below.



# Uniform Distribution

## Expectation and Variance

**Claim**

*If* $X \sim U(a,b)$, $\mathbb{E}(X) = \frac{1}{2}(a+b)$ *and* $V(X) = \frac{1}{12}(b-a)^2$.

**Proof.**

$$\begin{aligned}
\mathbb{E}(X) &= \int_a^b x \frac{1}{(b-a)} dx \\
&= \frac{1}{(b-a)} \frac{1}{2} x^2 \Big|_a^b \\
&= \frac{b^2 - a^2}{2(b-a)} \\
&= \frac{1}{2}(a+b)
\end{aligned}$$

$V(X)$ (Exercise) ∎

# Uniform Distribution

## Quickercise

Consider a random variable $X \sim U(2, 5)$. What is the probability that

- $X \geq 3$
- $3 \leq X \leq 4$

# Lecture Outline

## Exponential Distribution

# Exponential Distribution

## Motivating Example

Exponential distribution is commonly used to model time between consecutive events when the events occur according to Poisson distribution.

For example,

- The time duration between two accidents on a highway
- The amount of time taken by a bank teller to serve a customer
- The time between two arrivals at a checkout queue

# Exponential Distribution

## Probability Density Function

**Definition**

Suppose $X \sim \exp(\lambda)$, its probability density function is defined as

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- What is the support of this random variable
- Is this a valid PDF? i.e., is the area under this curve 1?
- What is its cumulative distribution function?

# Exponential Distribution

## Cumulative Density Function

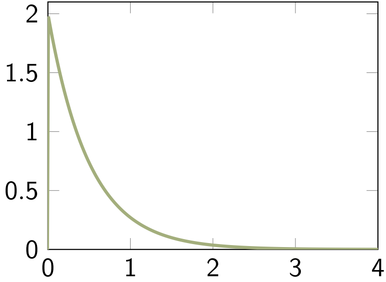
**Claim**

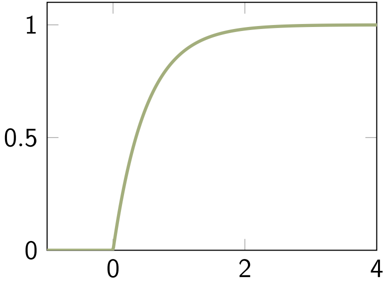
*Suppose* $X \sim \exp(\lambda)$, *its CDF is*

$$F_X(x) = 1 - e^{-\lambda x}$$

Check if $dF_X(x)/dx = f_X(x)$.

PDF and CDF of an exponentially distributed random variable with $\lambda = 2$ are shown below.



# Exponential Distribution

## Expectation and Variance

**Claim**

*If* $X \sim \exp(\lambda)$, *then* $\mathbb{E}(X) = 1/\lambda$ *and* $V(X) = 1/\lambda^2$

**Proof.**

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty x\lambda e^{-\lambda x} dx \\
&= x \int \lambda e^{-\lambda x} dx - \int \int \lambda e^{-\lambda x} dx \Big|_0^\infty \\
&= -xe^{-\lambda x} + \int e^{-\lambda x} dx \Big|_0^\infty \\
&= -xe^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty = \frac{1}{\lambda}
\end{aligned}$$

$V(X)$ (Exercise) ■

# Exponential Distribution

## Connections with Poisson Distribution

To see why inter-arrival times of an Poisson distributed random variable is exponentially distributed, let $X \sim \text{Pois}(\lambda)$.

Consider a time window $t$. The probability that there are zero arrivals in $t$ is given by

$$\mathbb{P}(X = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

If $Y$ is the inter-arrival time, then $\mathbb{P}(X = 0) = \mathbb{P}(Y > t)$.

Hence, $\mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$, which is the CDF of the exponential random variable.

# Exponential Distribution

## Quickercise

Suppose that the inter-arrival times of buses at a bus stop are exponentially distributed with rate $\lambda$. Let $X$ be the arrival time of the next bus.

Assuming, that you have been waiting for $t$ minutes (right after the passing of the previous bus), what is the probability that you will have to wait at least another $s$ minutes.

$$\mathbb{P}(X > s + t | X > t) =?$$

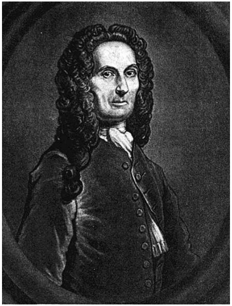
For this reason, exponential random variable is said to exhibit a **memory-less property**.

# Normal Distribution

# Normal Distribution

## Introduction

This is the most popular among all distributions. Some loose connections between the binomial theorem and normal distribution was discovered by De Moivre in early 1700s.



Gauss and Laplace are credited to have developed it further in their studies on least squares and the central limit theorem. Normal distribution is also commonly called as **Gaussian Distribution**.

# Normal Distribution

## Probability Density Function

**Definition**

The PDF of a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ with parameters $\mu$ and $\sigma^2$ is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- What is the support of this random variable
- Is this a valid PDF? i.e., is the area under this curve 1?
- What is its cumulative distribution function?

Mean and variance are called location and scale parameters. (Why?)

# Normal Distribution

## Cumulative Distribution Function

The normal distribution does not have a closed form CDF. Its CDF is often described using the **error function** that is defined as

$$\text{erf}(x) = \frac{1}{\pi} \int_{-x}^{x} e^{-z^2} dz$$

### Definition

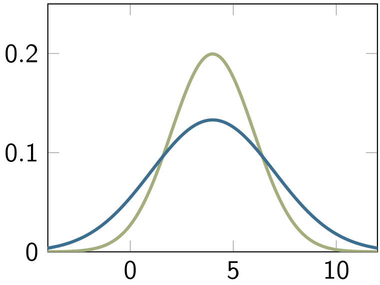
CDF of a Normal distributed random variable with parameters $\mu$ and $\sigma^2$ in terms of the erf function is

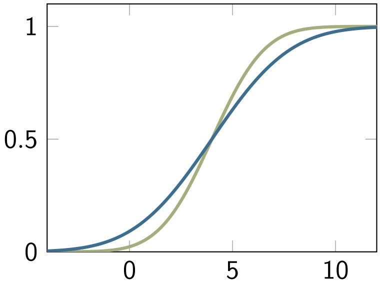
$$F_X(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x - \mu}{\sigma\sqrt{2}}\right)\right)$$

# Normal Distribution

## PMF and CDF

The PDF and CDF of a normally distributed random variable with mean 4 and standard deviation 2 (green) and 3 (blue) are shown below.



# Normal Distribution

## Standard Normal

Normal distribution with parameters 0 and 1 is very useful and is called the standard normal random variable.

**Definition**

The PDF of a standard normal random variable $Z \sim N(0, 1)$ with is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

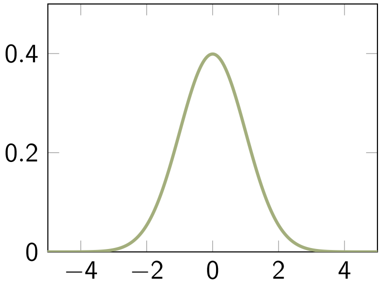
The symbol $\Phi(x)$ is commonly used to denote the CDF of the standard normal random variable.

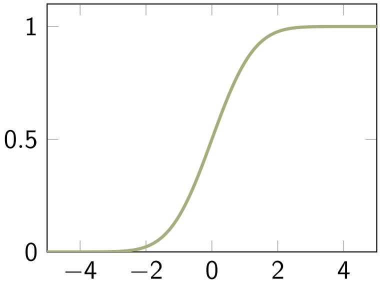
$$\Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$

# Normal Distribution

## PMF and CDF of Standard Normal

PDF and CDF of the standard normal random variable are shown below.



# Normal Distribution

## Expectation and Variance of Standard Normal Distribution

**Claim**

*Suppose* $Z \sim N(0,1)$, $\mathbb{E}(Z) = 0, V(Z) = 1$

**Proof.**

$$\begin{aligned}
\mathbb{E}(Z) &= \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \Big|_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

$V(X)$ (Exercise) ■

**Claim**

*Suppose* $X \sim N(\mu, \sigma^2)$, $\mathbb{E}(X) = \mu, V(X) = \sigma^2$

**Proof.**

Note that the standard normal and normal random variables are related as

$$X = \sigma Z + \mu$$

Hence, $\mathbb{E}(X) = \sigma\mathbb{E}(Z) + \mu = \mu$ and $V(X) = \sigma^2 V(Z) = \sigma^2$ ■

# Normal Distribution

## Reading Tables

Area $\Phi(x)$ Under the Standard Normal Curve to the Left of x

| x | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |

# Normal Distribution

## Quickercise

Consider a normally distributed random variable $X \sim \mathcal{N}(4, 9)$. What is the probability that

- $X \geq 5$
- $\mu - \sigma \leq X \leq \mu + \sigma$
- $\mu - 2\sigma \leq X \leq \mu + 2\sigma$

# Lecture Outline

## Other Distributions

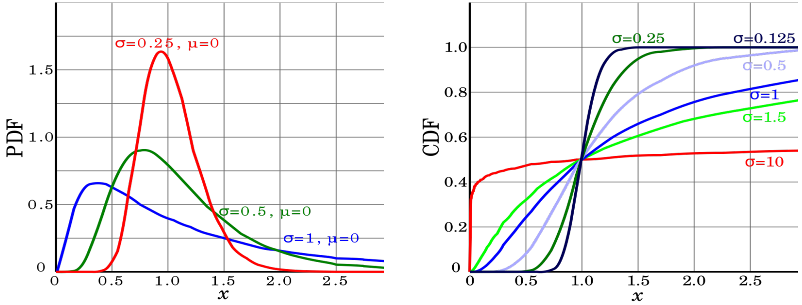
A random variable $X$ is said to be **log-normally** distributed or **Galton** distributed if $\ln X$ is normally distributed. In other words, $X = e^{\sigma+\mu Z}$, where $Z$ is a standard normal random variable.

| $X \sim \text{Lognormal}(\mu, \sigma^2)$ | |
|---|---|
| Parameters | $\mu \in \mathbb{R}, \sigma > 0$ |
| Support | $x \in (0, \infty)$ |
| PDF | $\dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right)$ |
| CDF | $\dfrac{1}{2} + \dfrac{1}{2}\text{erf}\left(\dfrac{\ln x - \mu}{\sigma\sqrt{2}}\right)$ |
| Expectation | $\exp\left(\mu + \dfrac{\sigma^2}{2}\right)$ |
| Variance | $\exp(2\mu + \sigma^2)\left(\exp(\sigma^2) - 1\right)$ |

Note that the parameters $\mu$ and $\sigma$ are the mean and standard deviation of $\ln X$ and not $X$.

# Other Distributions

## Lognormal Distribution



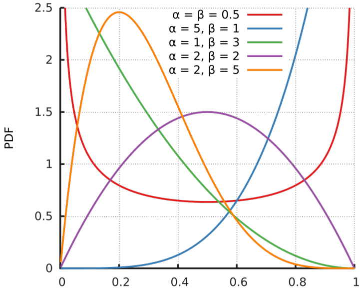
**Beta distributions** are common in situations in which the realizations of the random variable falls in an interval.

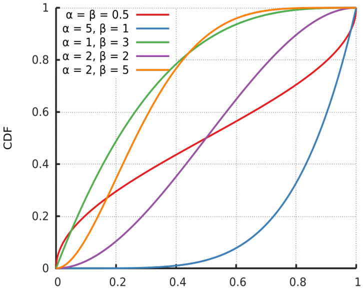
| $X \sim \text{Beta}(\alpha, \beta)$ | |
|---|---|
| Parameters | $\alpha > 0, \beta > 0$ |
| Support | $x \in (0, 1)$ |
| PDF | $\begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ |
| CDF | No closed form |
| Expectation | $\frac{\alpha}{\alpha+\beta}$ |
| Variance | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ |

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$

# Other Distributions

## Beta Distribution



# Other Distributions

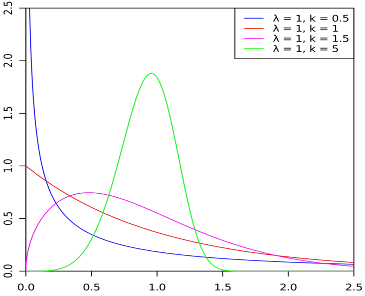
## Weibull Distribution

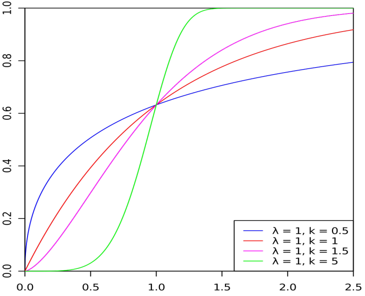
A commonly used distribution in reliability analysis is **Weibull distribution**. It is a more general version of the exponential random variable.

| $X \sim \text{Weibull}(\alpha, \beta)$ | |
|---|---|
| Parameters | $\alpha > 0, \beta > 0$ |
| Support | $x \in [0, \infty)$ |
| PDF | $\begin{cases} \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ |
| CDF | $\begin{cases} 1 - e^{-(x/\alpha)^{\beta}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Expectation | $\alpha\Gamma(1 + 1/\beta)$ |
| Variance | $\alpha^2\left(\Gamma(1 + 2/\beta) - (\Gamma(1 + 1/\beta))^2\right)$ |

# Other Distributions

## Weibull Distribution



# Other Distributions

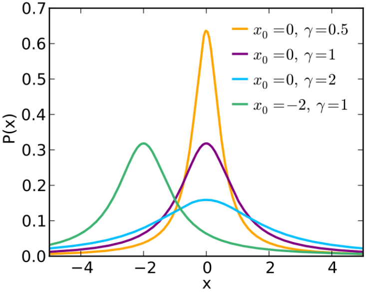
## Cauchy Distribution

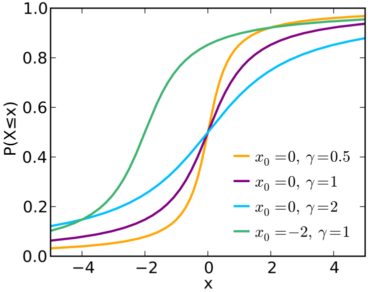
**Cauchy distributed random variables** are interesting because they do not have a finite mean or variance! (Just like the random variable in St. Petersburg paradox)

| $X \sim \text{Cauchy}(\theta, \alpha)$ | |
|---|---|
| Parameters | $\theta \in \mathbb{R}, \alpha > 0$ |
| Support | $x \in \mathbb{R}$ |
| PDF | $\dfrac{1}{\pi\alpha\left(1 + ((x-\theta)/\beta)^2\right)}$ |
| CDF | $\dfrac{1}{\pi}\tan^{-1}\left(\dfrac{x-\theta}{\alpha}\right) + \dfrac{1}{2}$ |

# Other Distributions

## Cauchy Distribution



# Other Distributions

## Gamma Distribution

**Gamma distributed** random variables are used to model the time until occurrence of $\alpha$ events.

| $X \sim \Gamma(\alpha, \beta)$ | |
|---|---|
| Parameters | $\alpha > 0, \beta > 0$ |
| Support | $x \in (0, \infty)$ |
| PDF | $\begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ |
| CDF | No simple closed form |
| Expectation | $\alpha/\beta$ |
| Variance | $\alpha/\beta^2$ |

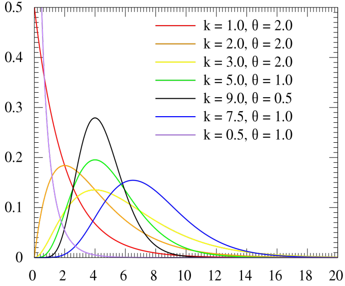
where $\Gamma(\alpha)$ is called the Gamma function and is defined as

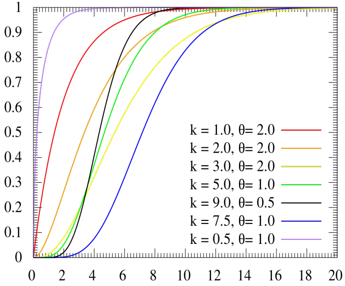
$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$$

This function is just like the factorial but is also defined for non-integers and satisfies $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

# Other Distributions

## Gamma Distribution




A special case of Gamma distribution is called **Chi-square** or $\chi_n^2$ distributed with $n$ degrees of freedom.

# A Solution

# A Solution

## Prospect Theory

Expected value is the same in both cases. But then one is positive and the other is negative.

A majority of you were risk-averse in the first case and risk-seeking in the second case.

Daniel Kahneman and Amos Tversky in 1979 highlighted the pitfalls of using the expectation of utilities. Kahneman won the Nobel in 2002 for their work on prospect theory.

Framing effects and the way we perceive gains and losses play a major role in human decision making.

# Your Moment of Zen