CE 211
Mathematics for Engineers

Lecture 1
Introduction

Lecture Outline

Lecture Outline

The Problem

The Problem

The Birthday Problem

In a class of n students, what is the probability that two or more students share the same birthday?

Lecture Outline

Course Overview

Course Overview

Frequently Asked Questions (FAQs)

Why take this course?

- (CE students) It is a core course. You have no choice :)
- (Others) If your research requires some mathematical background, this course covers material on commonly used topics.

What will I learn from this part of the course?

- We will mostly discuss applied probability which deals with how uncertainty can be modeled and understood.
- A couple of lectures towards the end are dedicated for introductory statistics.

Course Overview

Prerequisites and Texts

While the course does not have assume prerequisites background in elementary calculus is assumed.

The following books can be used as references for this course:

1. Ross, S. (2014). A first course in probability. Pearson.
2. Papoulis, A., & Pillai, S. U. (2002). Probability, random variables and stochastic processes. Tata McGraw Hill.

*Other editions of these books can also be used

Microsoft Teams will be used for course communication. Lecture slides will be posted on Teams in advance. Make sure to skim through them before coming to the class.

Course Overview

Assignments, Exams, and Grading

Written Assignments

- This part of the course will have 2 written assignments.
- You are encouraged to discuss the problems but **you must submit your own solutions**.
- Plagiarism is strictly prohibited and will be penalized.

Examinations

- There will be online quizzes for this part of the course.
- End-semester exam is comprehensive

Grading TBD

Course Overview

Course Feedback

- At the end of each class, you are required to provide feedback on Teams by answering if 'the contents of the lecture were clear and easy to understand?'
- Responses are to be provided on a Likert scale (Strongly disagree, Disagree, Neither, Agree, Strongly agree).
- These stats will help me calibrate the course content and also in picking the right questions for the assignments.

Course Overview

Office Hours

- If you have any course related queries, feel free to post a note on Teams or schedule an appointment by email.
- While sending emails regarding the course, please include "CE-211-2020" in the subject line. This will make it easier for me to track mails.
- Attendance is a must unless you have internet issues or other valid reasons.

Combinatorics

Combinatorics

Permutations

Definition (Permutations)

The number of ways of arranging n items is $n!$

There are n ways to select the first item For each choice of the first item, there are $n-1$ ways to select the second, and so on.

For example, the permutations of three letters A, B, and C are ABC, ACB, BAC, BCA, CAB, CBA. Here, the order of arrangements are important.

Definition (k-Permutations)

The number of ways of arranging k of n items is given by $\dfrac{n!}{(n-k)!}$

Definition (Combinations)

The number of ways of selecting k items from a set of n items is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Here, the order of arrangements is not important. For instance, the number of combinations of two of the three letters A, B, and C are $\{$A,B$\}$, $\{$B,C$\}$, and $\{$C,A$\}$.

Combinatorics

Quickercise

- In how many unique ways can n people be circularly arranged?
- In how many ways can your rearrange the letters of the word PROBABILITY?
- Find the number of ways of dividing 22 cricket players into two teams of 11 each?
- A standard deck of 52 cards has an equal number of $\heartsuit$, $\clubsuit$, $\diamondsuit$, and $\spadesuit$. In how many ways can we arrange them such that all cards of the same suit are next to each other.
- You are creating a play list with 5 rap songs and 5 classical pieces. In how many ways can you do this without having songs of the same genre next to each other.

Prove the **Binomial Theorem:** Let n be a positive integer.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Hence, $\binom{n}{k}$ is also called **Binomial Coefficients**. Using the above theorem, show that

- $\binom{n}{k} = \binom{n}{n-k}$
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Provide an alternate proof for the above statements using a mathematical or a combinatorial argument.

Combinatorics

Pascal's Triangle

Pascal Triangle: For integers $1 \le k \le n-1$, show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Combinatorics

Partitions

The number of ways of partitioning n items into r groups with each group having n_1, n_2, ..., n_r items (which add up to n) is given by

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\dots\binom{n-\sum_{k=1}^{r-1} n_k}{n_r} = \frac{n!}{n_1!n_2!\cdots n_r!} = \binom{n}{n_1, n_2, \dots, n_r}$$

Can you derive the same answer using permutations?

Multinomial theorem: Let n be a positive integer

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{n_1 + \ldots + n_2 = n} \frac{n!}{n_1! n_2! \cdots n_r!} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

For this reason, $\binom{n}{n_1, n_2, \ldots, n_r}$ are also called **multinomial coefficients**.

Using a combinatorial argument, show that

$$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_r} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_r} + \ldots + \binom{n-1}{n_1, n_2, \ldots, n_r - 1}$$

Given an integer n, how many non-negative solutions exist for the equation $x_1 + x_2 + \ldots + x_r = n$.

Imagine n zeros and $(r-1)$ ones. The ones serve as markers which split the zeros into $x_1, x_2, \ldots, x_r$. For example, when $n = 10$ and $r = 5$, a feasible solution can be expressed as

$$10001010010000$$

Thus, of the $n + r - 1$ spaces, we need to choose $r - 1$ spots. This can be done in $\binom{n+r-1}{r-1}$ ways. What if the xs have to be strictly positive?

Combinatorics

Challenges

Combinatorial problems can be challenging since one can arrive at the answer in several different ways and there might be no set procedure to solve the problem.

The tools that we have seen so far are handy when dealing with counting, but problems on this topic often require more creativity. Let's take a look at a couple of examples.

Lemma (Handshaking Lemma)

At a gathering of n individuals, assume that some shake hands (no two individuals repeat) and no one shakes his/her own hand. Show that the number of people who shake hands an odd number of times is even.

The *double counting* proof technique is nothing but an obvious fact. If you can count the number of ways in which some event happens in two ways, the answers must be the same.

For example, if tasks A and B can be performed in n and m ways respectively. The number of ways to perform both tasks (assuming order of tasks is not important) is $nm = mn$.

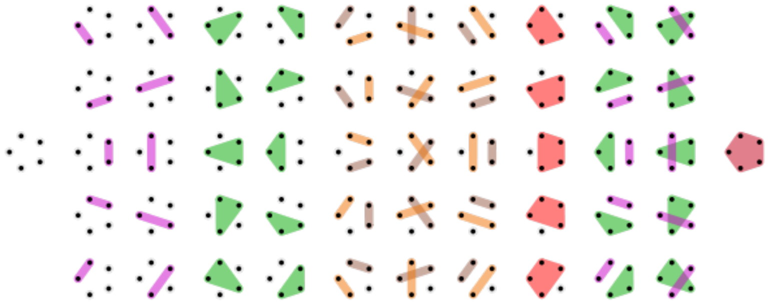
We have already used this in many of the combinatorial arguments discussed thus far. But the handshaking lemma is a more crafty application.

Suppose a set A has n elements. Define B_n as the number of elements of the set containing all partitions of n. For example, suppose $A = \{1, 2, 3\}$. Then, $B_3 = 5$ since the set of all partitions is

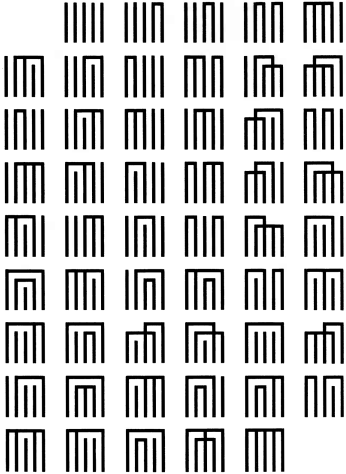
$$\big\{\{1\}, \{2\}, \{3\}\big\}$$
$$\big\{\{1, 2\}, \{3\}\big\}$$
$$\big\{\{1\}, \{2, 3\}\big\}$$
$$\big\{\{1, 3\}, \{2\}\big\}$$
$$\big\{\{1, 3, 2\}\big\}$$

How many elements are present in B_n for a given n?

Here is a graphic which illustrates the set of all partitions for $n = 5$.



From The Tale of Genji, a Japanese novel from the 11th century,



In problems like, these, it is helpful to build the solution from smaller values of n using recursion.

The values of B_n, also called **Bell numbers**, satisfy the following recursive relationship.

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

To see why, consider partitions in which the $(n+1)$th element appears as a singleton. The number of such partitions is B_n.

Now consider the case where the $(n+1)$th element appears along with exactly one of the remaining n elements. The other element can be selected in $\binom{n}{1}$ ways and the remaining elements can be partitioned in B_{n-1} ways.

Proceeding similarly, we get

$$B_{n+1} = \binom{n}{0} B_n + \binom{n}{1} B_{n-1} + \binom{n}{2} B_{n-2} + \ldots + \binom{n}{n} B_0 = \sum_{k=0}^{n} \binom{n}{k} B_k$$

The Bell numbers can also be derived iteratively using a Bell triangle.

1					
1	2				
2	3	5			
5	7	10	15		
15	20	27	37	52	
52	67	87	114	151	203

Sets and Probability

A set is a collection of elements (could be numbers, words, buildings, songs, functions, matrices, etc.). We write $x \in A$ to indicate that x belongs to A and $x \notin A$ otherwise. An **empty** set is represented using $\emptyset$. **Complement** of a set A will be denoted by A^c or A'. **Power set** is written as 2^A.

If every element of A is in B, we say A is a **subset** of B and write $A \subset B$. Likewise, we say B is a **superset** of A and write $B \supset A$.

The **union** of two sets A and B is defined as the set of elements which are in either A or B or both. Mathematically,

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$$

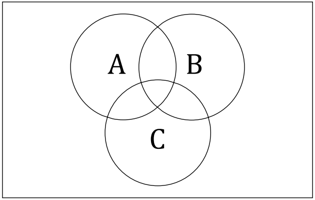
The **intersection** of two sets A and B is defined as the set of elements that are in both A and B. That is,

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$$

Two sets A and B are said to be disjoint if $A \cap B = \emptyset$.

Several set-related equalities can be derived for unions and intersections. Venn diagrams is one way of proving them.

- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$
- $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
- $(A \cap B \cap C)^c = A^c \cup B^c \cup C^c$



The last two equations are also called **De Morgan's laws**.

Sets and Probability

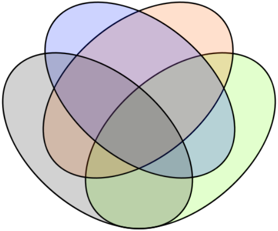
Venn Diagrams

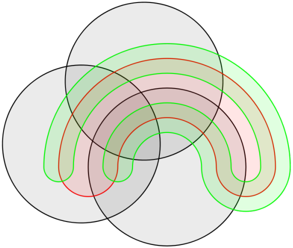
Venn diagrams can also help visualize identities for **finite** sets such as

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

More generally, $|A_1 \cup \cdots \cup A_n| = \sum_{I \subset \{1,\ldots,n\}} (-1)^{|I|+1} | \cap_{i \in I} A_i|$





Let us now look at some simple sets and study their cardinatity, i.e., let's count the number of elements in them.

Consider a set A comprising of the all continents in the world. Such sets which have a finite number of elements are called **finite** sets.

On the other hand, consider the set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. This is clearly not a finite set and there are an infinite number of elements in them.

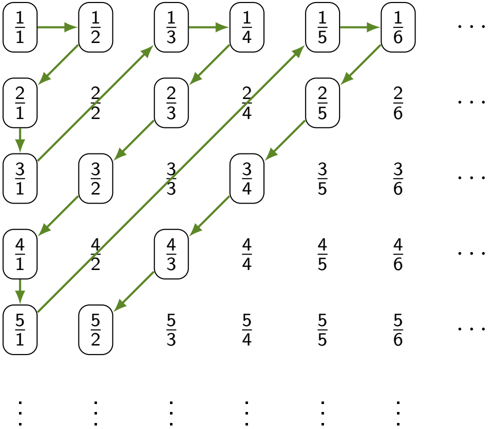
How about the set of even integers $\{\ldots, -2, 0, 2, \ldots\}$? Which of the two infinite sets is bigger? How about the set of integer coordinates on a 2D plane, i.e., $\{(x, y) | x \in \mathbb{Z}, y \in \mathbb{Z}\}$?

Definition (Countably Infinite)

A set A is countably infinite if there exists a bijective function f from A to the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots,\}$.

Sets that are finite or countably infinite are called countable. The foundations for counting sets were laid by Georg Cantor in the late 1800s.

The set of rational numbers $\mathbb{Q}$ is also countably infinite!



Sets and Probability

Uncountable Case

Sets that are not countable are called **uncountable**. The set of real numbers $\mathbb{R}$ is uncountable since we cannot create a bijective function to $\mathbb{N}$. Cantor proved this using a diagonalization argument.

Let's proceed by contradiction. We will show that the unit interval $(0, 1)$ is uncountable. Extension to $\mathbb{R}$ is trivial. (Why?) Suppose we can list all reals in $(0, 1)$ and associate each number with a natural number

$$\begin{array}{ll}
1 & 0.\mathbf{3}895127\ldots \\
2 & 0.2\mathbf{5}00000\ldots \\
3 & 0.62\mathbf{4}6346\ldots \\
4 & 0.222\mathbf{2}222\ldots \\
5 & 0.1225\mathbf{7}43\ldots \\
6 & 0.58521\mathbf{5}8\ldots \\
\vdots & \quad\vdots
\end{array}$$

Construct a number which differs from the diagonal elements shown in bold. E.g., 0.472561...This will never appear in the above list!

Sets and Probability

Some Infinities are Bigger than Others

To summarize, sets that are countable are either finite or countably infinite such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.

On the other hand, there are several sets such as the set of irrationals $\mathbb{Q}^c$ and $\mathbb{R}$ which are uncountable. One can also show bijections between $\mathbb{R}$, $\mathbb{R}^n$, and even $2^{\mathbb{N}}$!

These results have serious implications in the way we study probability from a theoretical standpoint.

Sets and Probability

Understanding Probability

There are multiple ways in which the term probability can be interpreted. For example,

- Suppose an unbiased coin is being tossed. The probability of observing heads is 0.5. One way to interpret 0.5 is that it is the frequency of occurrence of heads when we conduct a large number of trials.
- On the other hand, consider a statement "Based on historical records, there is an 80% chance that Subhas Chandra Bose died in a plane crash". The frequency argument won't work here since there is no repetition. In such instances, probability can be viewed as a subjective belief.

Sets and Probability

Finite Case

Let us now consider some simple examples and define a framework for studying probability.

Suppose an unbiased coin is tossed thrice. This act is what we call an **experiment**. Each experiment results in an **outcome**, e.g., Heads, Tails, Heads.

The set of all possible outcomes is called the **sample space** which is usually denoted by Ω.

Specify the sample space in the following experiment

- Tossing a coin thrice
- Rolling two dice

Sets and Probability

Finite Case

We are usually interested in the probability that one or some of the outcomes the sample space occur.

These questions can be translated to a subset of outcomes that are called **events**. We say that an event happened if one of the outcomes in the event occurs during the experiment.

For example, in the previous experiments

- What is the event where we see exactly two heads

$$\{HHT, HTH, THH\}$$

- What is event where the sum of the dice is 7

$$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

Note that not all subsets can be easily described in words. But we still treat them as events and can ask the probability of its occurrence.

An intuitive and familiar way for computing probabilities of events is to look at the number of elements in the event and divide it by the total number of outcomes.

Definition (Discrete Uniform Probability)

Suppose the sample space of an experiment consists of n outcomes which are equally likely, then the probability of an event A is

$$\mathbb{P}(A) = \frac{|A|}{n}$$

One must be careful in constructing the outcomes of the sample space. For example, when two dice are thrown, the sum has 11 outcomes: 2, 3, . . . , 12. Using this argument, the probability is 1/11. If you treat the dice to be indistinguishable, the answer would be 3/21 (Why?) Both the answers are wrong because all outcomes are not equally likely.

Sets and Probability

Finite Case

This notion of probability can solve several practical problems. Results regarding the cardinatilty of sets extend naturally. For example,

- $\mathbb{P}(\Omega) = 1, \mathbb{P}(\emptyset) = 0$
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{I \subset \{1,\dots,n\}} (-1)^{|I|+1} \mathbb{P}(\cap_{i \in I} A_i)$
- $\mathbb{P}\left((\cup_{i=1}^n A_i)^c\right) = \mathbb{P}(\cap_{i=1}^n A_i^c)$

Sets and Probability

Finite Case

To summarize, when we have a finite number of equally likely outcomes, the framework for studying probability involves

- A sample space Ω consisting of all outcomes
- The set of all events 2^{Ω}
- A probability function $\mathbb{P} : 2^{\Omega} \to [0, 1]$

However, things get complicated when the sample space is countably infinite or uncountable. For example,

- Suppose we randomly pick an integer. What is the probability that it is even?
- Imagine throwing darts on the unit interval $[0, 1]$. What is the probability that we hit $1/4$? What is the probability that we hit a rational number?

Lecture Outline

A Solution

A Solution

The Birthday Problem

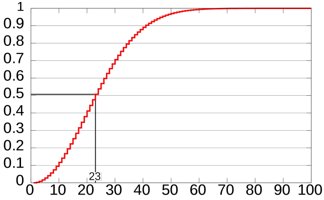
Ignore leap years and assume that there are 365 days and let A be the event that at least two people have the same birthday. The sample space Ω is the set of all possible birthdays of everyone and has 365^n elements.

Let's calculate the probability of A^c, the probability that no two have the same birthday. In a class of 60 individuals, that would be

$$\mathbb{P}(A^c) = \frac{365}{365} \times \frac{364}{365} \times \ldots \times \frac{306}{365}$$

Therefore,

$$\mathbb{P}(A) = 1 - \frac{365}{365} \times \frac{364}{365} \times \ldots \times \frac{306}{365} = 0.994$$



Your Moment of Zen