CE 205A
Transportation Logistics

Lecture 2
Linear Programming Review - Part II

Previously on Transportation Logistics

Expanding the constraints we can write $\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$. Hence, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$. Writing the objective function in terms of the non-basic variables

$$
\begin{aligned}
z &= \mathbf{c}_B^\mathsf{T}\mathbf{x}_B + \mathbf{c}_N^\mathsf{T}\mathbf{x}_N \\
&= \mathbf{c}_B^\mathsf{T}(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N) + \mathbf{c}_N^\mathsf{T}\mathbf{x}_N \\
&= \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^\mathsf{T} - \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N
\end{aligned}
$$

Theorem

Suppose $\mathbf{x}^*$ is a basic feasible solution and $\mathbf{c}^\mathsf{T} - \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{A} \geq 0^\mathsf{T}$, then $\mathbf{x}^*$ is optimal

Previously on Transportation Logistics

Can you compute $\mathbf{c}^\mathsf{T}\mathbf{d}$ when we move in the direction that increases the non-basic variable x_j?

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_B \\ \mathbf{d}_N \end{bmatrix} = \begin{bmatrix} \mathbf{d}_B \\ \mathbf{e}_j \end{bmatrix}$$

$$\mathbf{c}^\mathsf{T}\mathbf{d} = \begin{bmatrix} \mathbf{c}_B^\mathsf{T} & \mathbf{c}_N^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{d}_B \\ \mathbf{e}_j \end{bmatrix} = \mathbf{c}_B^\mathsf{T}\mathbf{d}_B + \mathbf{c}_N^\mathsf{T}\mathbf{e}_j = \mathbf{c}_B^\mathsf{T}\mathbf{d}_B + c_j$$

Recall, that the reduced cost of non-basic variable x_j is $c_j - \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{A}_{\cdot j}$ and if we are not at an optimal solution, we may find a j such that $\bar{c}_j < 0$*

Hence, we pick a direction vector

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_B \\ \mathbf{d}_N \end{bmatrix} = \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{A}_{\cdot j} \\ \mathbf{e}_j \end{bmatrix}$$

Previously on Transportation Logistics

We have a basic feasible solution/corner point and a direction. The immediate question is how far should we move along this direction.

Too small a step size and we will be close to where we started. Too big and we may overshoot the feasible region. We want the right step size to help us land at the extreme point.

Let's find the largest step size which will keep the new point $\hat{\mathbf{x}}$ feasible. Hence, the following conditions must be true

$$\mathbf{A}(\mathbf{x} + \eta \mathbf{d}) = \mathbf{b} \qquad \text{(Condition 1)}$$
$$\mathbf{x} + \eta \mathbf{d} \geq \mathbf{0} \qquad \text{(Condition 2)}$$

Previously on Transportation Logistics

1. Start with an initial basic feasible solution $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$

2. Compute the reduced cost vector $\bar{\mathbf{c}}_N^\mathsf{T} = \mathbf{c}_N^\mathsf{T} - \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{N}$

3. If $\bar{\mathbf{c}}_N \geq 0$, then $\mathbf{x}$ is optimal and terminate, else go to Step 4

4. Pick $j^* : \bar{c}_{j^*} < 0$ and compute descent direction $\mathbf{d} = \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{A}_{\cdot j^*} \\ \mathbf{e}_{j^*} \end{bmatrix}$

5. If $\mathbf{d}_B \geq 0$, then the LP is unbounded, else go to Step 6

6. Set $k^* \in \arg\min \left\{ -\frac{(x_B)_k}{(d_B)_k} : (d_B)_k < 0 \right\}$

7. Modify the basis by swapping $\mathbf{B}_{\cdot k^*}$ and $\mathbf{A}_{\cdot j^*}$, set $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, and go to Step 2.

Lecture Outline

1. Simplex with Bounds
2. Duality
3. Sensitivity Analysis
4. CPLEX

Lecture Outline

Simplex with Bounds

Simplex with Bounds

Introduction

Consider the linear program with upper and lower bounds

$$\min \mathbf{c}^\mathsf{T}\mathbf{x}$$
$$\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\ell \leq \mathbf{x} \leq \mathbf{u}$$

The bounds can be treated as regular constraints and slack and surplus variables can be added. What is the size of the number of constraints and variables? $m + 2n$ constraints and $3n$ variables.

How does the feasible region for this problem look like? Can basic solutions and basic feasible solutions be derived from columns of $\mathbf{A}$? Are basic feasible solutions extreme points?

Simplex with Bounds

Introduction

Sketch the feasible region and find all basic feasible solutions to the following system of inequalities.

$$x_1 + x_2 \leq 5$$
$$-x_1 + 2x_2 \leq 4$$
$$0 \leq x_1 \leq 4$$
$$-1 \leq x_2 \leq 4$$

▶ Add slacks to the first two inequalities.

▶ Select two basic columns and write them in terms of the non-basic variables.

▶ Fix the non-basic variables at their upper and lower bounds to get basic feasible solutions.

Simplex with Bounds

Consider the following expanded version of the LP

$$\min \ \mathbf{c}_B^\mathsf{T}\mathbf{x}_B + \mathbf{c}_{N_l}^\mathsf{T}\mathbf{x}_{N_l} + +\mathbf{c}_{N_u}^\mathsf{T}\mathbf{x}_{N_u}$$

$$\text{s.t.} \ \begin{bmatrix} \mathbf{B} & \mathbf{N}_l & \mathbf{N}_u \end{bmatrix} \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_{N_l} \\ \mathbf{x}_{N_u} \end{bmatrix} = \mathbf{b}$$

$$\boldsymbol{\ell}_B \leq \mathbf{x}_B \leq \mathbf{u}_B$$

$$\boldsymbol{\ell}_{N_l} \leq \mathbf{x}_{N_l} \leq \mathbf{u}_{N_l}$$

$$\boldsymbol{\ell}_{N_u} \leq \mathbf{x}_{N_u} \leq \mathbf{u}_{N_u}$$

Just as before, we solve for $\mathbf{x}_B$ by setting $\mathbf{x}_{N_l} = \boldsymbol{\ell}_{N_l}$ and $\mathbf{x}_{N_u} = \mathbf{u}_{N_u}$ to get

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}_l\boldsymbol{\ell}_{N_l} - \mathbf{B}^{-1}\mathbf{N}_u\mathbf{u}_{N_u}$$

This represents a basic solution. If it is also feasible, that is if $\boldsymbol{\ell}_B \leq \mathbf{x}_B \leq \mathbf{u}_B$, then it is a basic feasible solution.

Simplex with Bounds

Optimality Conditions

Given $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}_l\mathbf{x}_{N_l} - \mathbf{B}^{-1}\mathbf{N}_u\mathbf{x}_{N_u}$, the objective can be written as

$$z = \mathbf{c}_B^{\mathsf{T}}(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}_l\mathbf{x}_{N_l} - \mathbf{B}^{-1}\mathbf{N}_u\mathbf{x}_{N_u}) + \mathbf{c}_{N_l}^{\mathsf{T}}\mathbf{x}_{N_l} + \mathbf{c}_{N_u}^{\mathsf{T}}\mathbf{x}_{N_u}$$
$$= \mathbf{c}_B^{\mathsf{T}}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_{N_l}^{\mathsf{T}} - \mathbf{c}_B^{\mathsf{T}}\mathbf{B}^{-1}\mathbf{N}_l)\mathbf{x}_{N_l} + (\mathbf{c}_{N_u}^{\mathsf{T}} - \mathbf{c}_B^{\mathsf{T}}\mathbf{B}^{-1}\mathbf{N}_u)\mathbf{x}_{N_u}$$

If $\mathbf{c}_{N_l}^{\mathsf{T}} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N}_l \geq \mathbf{0}^{\mathsf{T}}$, increasing the $\mathbf{x}_{N_l}$ values which are at their lower bounds will only increase the objective.

If $\mathbf{c}_{N_u}^{\mathsf{T}} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N}_u \leq 0$, decreasing $\mathbf{x}_{N_u}$ will again increase the objective.

Theorem

Suppose $\mathbf{x}^*$ is a basic feasible solution and $\mathbf{c}_{N_l}^{\mathsf{T}} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N}_l \geq \mathbf{0}^{\mathsf{T}}$ and $\mathbf{c}_{N_u}^{\mathsf{T}} - \mathbf{c}_B\mathbf{B}^{-1}\mathbf{N}_u \leq \mathbf{0}^{\mathsf{T}}$, then $\mathbf{x}^*$ is optimal

What if the optimality conditions are violated for multiple non-basic variables? Which variable do you choose?

Simplex with Bounds

If the optimality conditions are violated, then we can find a search direction $\mathbf{d}$ that improves the objective. Suppose there is a variable j at its lower bound for which $c_j - \mathbf{c}_B \mathbf{B}^{-1} \mathbf{A}_{\cdot j} < 0$. Then, the following vector is a descent direction. (Why?)

$$\mathbf{d} = \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{A}_{\cdot j} \\ \mathbf{e}_j \\ \mathbf{0} \end{bmatrix}$$

Likewise, if a non-basic variable is at its upper bound, then it is easy to show that the descent direction takes the form

$$\mathbf{d} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{A}_{\cdot j} \\ \mathbf{0} \\ -\mathbf{e}_j \end{bmatrix}$$

Simplex with Bounds

The step sizes can be computed just as before by trying to maintain feasibility. Suppose the entering variable was at its lower bound.

Condition 1: $\mathbf{A}(\mathbf{x} + \eta\mathbf{d}) = \mathbf{b}$. This is automatically satisfied because

$$\begin{bmatrix} \mathbf{B} & \mathbf{N}_l & \mathbf{N}_u \end{bmatrix} \begin{bmatrix} \mathbf{x}_B - \eta\mathbf{B}^{-1}\mathbf{A}_{.j} \\ \boldsymbol{\ell}_{N_l} + \eta\mathbf{e}_j \\ \mathbf{u}_{N_u} + \eta\mathbf{0} \end{bmatrix} = \mathbf{b} - \eta\mathbf{B}\mathbf{B}^{-1}\mathbf{A}_{.j} + \eta\mathbf{N}_l\mathbf{e}_j$$

$$= \mathbf{b}$$

Condition 2: Three sub-cases are possible

$$\begin{bmatrix} \boldsymbol{\ell}_B \\ \boldsymbol{\ell}_{N_l} \\ \boldsymbol{\ell}_{N_u} \end{bmatrix} \leq \begin{bmatrix} \mathbf{x}_B \\ \boldsymbol{\ell}_{N_l} \\ \mathbf{u}_{N_u} \end{bmatrix} + \eta \begin{bmatrix} \mathbf{d}_B \\ \mathbf{e}_j \\ \mathbf{0} \end{bmatrix} \leq \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_{N_l} \\ \mathbf{u}_{N_u} \end{bmatrix}$$

Simplex with Bounds

Condition 2(a): $\ell_B \leq \mathbf{x}_B + \eta \mathbf{d}_B$

Writing it component wise, $\eta(d_B)_k \geq -(x_B - \ell_B)_k$. If $(d_B)_k \geq 0$, this inequality is true for all η. Else, we need

$$\eta \leq -\frac{(x_B - \ell_B)_k}{(d_B)_k}$$

Hence, we set

$$\eta = \min\left\{-\frac{(x_B - \ell_B)_k}{(d_B)_k} : (d_B)_k < 0\right\}$$

Simplex with Bounds

Condition 2(b): $\mathbf{x}_B + \eta \mathbf{d}_B \leq \mathbf{u}_B$

If $(d_B)_k \leq 0$, the above inequality is automatically satisfied. If $(d_B)_k > 0$, $\eta(d_B)_k \leq (u_B - x_B)_k$, implies

$$\eta \leq \frac{(u_B - x_B)_k}{(d_B)_k}$$

Hence, we set

$$\eta = \min\left\{ \frac{(u_B - x_B)_k}{(d_B)_k} : (d_B)_k > 0 \right\}$$

Simplex with Bounds

Condition 2(c): $\ell_{N_l} + \eta \mathbf{e}_j \leq \mathbf{u}_{N_l}$. This implies that $\eta \leq u_j - l_j$.

Combining all three conditions, the step-size allowed or the min-ratio test can be summarized as

$$\eta = \min \begin{cases} \min \left\{ -\frac{(x_B - \ell_B)_k}{(d_B)_k} : (d_B)_k < 0 \right\} \\ \min \left\{ \frac{(u_B - x_B)_k}{(d_B)_k} : (d_B)_k > 0 \right\} \\ u_j - l_j \end{cases}$$

A similar exercise can be followed for reducing a non-basic variable at its upper bound.

Lecture Outline

Duality

Duality

Consider the problem of optimizing nutritional requirements from various food types. Find the quantity of each food type to purchase to meet the minimum daily requirements (MDR) from different quantities of food while minimizing the total cost.

	Carbohydrates	Protein	Calcium	Iron	Unit Price
Milk	6.1	310	10.5	18	11
Wheat	44.7	1411	2	365	36
Rice	21.2	460	0.6	41	7.5
Sugar	34.9	0	0	0	51.7
Potatoes	14.3	336	1.8	118	34
Spinach	1.1	106	0	138	8.1
MDR	3000	70	0.8	12	

Let x_j be the amount of food type to purchase. The optimization problem for minimizing the cost while meeting MDR can be written as

$$\min \ 11x_1 + 36x_2 + 7.5x_3 + 51.7x_4 + 34x_5 + 8.1x_6$$
$$\text{s.t. } 6.1x_1 + 44.7x_2 + 21.2x_3 + 34.9x_4 + 14.3x_5 + 1.1x_6 \geq 3000$$
$$310x_1 + 1411x_2 + 460x_3 + 0x_4 + 336x_5 + 106x_6 \geq 70$$
$$10.5x_1 + 2x_2 + 0.6x_3 + 0x_4 + 1.8x_5 + 0x_6 \geq 0.8$$
$$18x_1 + 365x_2 + 41x_3 + 0x_4 + 118x_5 + 138x_6 \geq 12$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Imagine that Elon Musk that has synthesized these nutrients in the lab and is planning to sell them as tablets that provide 1 unit of carbohydrates, protein, etc. How should he set their prices of each tablet?

Duality

Let $y_1, y_2, y_3,$ and y_4 be the cost of each tablet. Elon would obviously like to maximize his revenue from the sales. Hence, an ideal objective would be

$$\max\ 3000y_1 + 70y_2 + 0.8y_3 + 12y_4$$

Note that this model doesn't require a minimum of 3000 units of Carbohydrates because that it is not a constraint for his company. His decision variable is the price and we will assume that he is capable of meeting demand for his tablets.

The prices must be *fair* and competitive to what is available out there.

Duality

The new prices must be cheaper in terms of the benefits for each of the food types. For instance, to produce the benefits from 1 unit of Milk, we would need to spend $6.1y_1 + 310y_2 + 10.5y_3 + 18y_4$ on the tablets. Hence,

$$6.1y_1 + 310y_2 + 10.5y_3 + 18y_4 \leq 11$$

Thus, we can write a new optimization problem for the tablet manufacturer as

$$
\begin{aligned}
\max\ & 3000y_1 + 70y_2 + 0.8y_3 + 12y_4 \\
\text{s.t.}\ & 6.1y_1 + 310y_2 + 10.5y_3 + 18y_4 \leq 11 \\
& 44.7y_1 + 1411y_2 + 2y_3 + 365y_4 \leq 36 \\
& 21.2y_1 + 460y_2 + 0.6y_3 + 41y_4 \leq 7.5 \\
& 34.9y_1 + 0y_2 + 0y_3 + 0y_4 \leq 51.7 \\
& 14.3y_1 + 336y_2 + 1.8y_3 + y_4118 \leq 134 \\
& 1.1y_1 + 106y_2 + 0y_3 + 138y_4 \leq 8.1 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

Duality

Given a LP, a closely related formulation called the dual LP can be written as follows:

Primal LP	**Dual LP**
min $\mathbf{c}^\mathsf{T}\mathbf{x}$	max $\mathbf{b}^\mathsf{T}\mathbf{y}$
s.t. $\mathbf{Ax} \geq \mathbf{b}$	s.t. $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \geq \mathbf{0}$

The dual has the following features:

▶ If the primal is a minimization problem, the dual has a maximization objective. The dual of the dual is the primal.

▶ The dual has as many variables (constraints) as the constraints (variables) of the primal

▶ The dual variables are also called *shadow prices*. The name comes from economics where a new good may not have a market price and the willingness to pay is treated as a shadow price.

What if the primal looks a bit different? Can you write the dual of the following optimization problem:

$$\min \ \mathbf{c}^\mathsf{T} \mathbf{x}$$
$$\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

The dual of this problem is

$$\max \ \mathbf{b}^\mathsf{T} \mathbf{y}$$
$$\text{s.t. } \mathbf{A}^\mathsf{T} \mathbf{y} \leq \mathbf{c}$$

If we had an equality constraint in the primal problem, the corresponding dual variable will be unrestricted.

Duality

min problem		max problem
ith constraint $\geq$	$\leftrightarrow$	ith variable ≥ 0
ith constraint $\leq$	$\leftrightarrow$	ith variable ≤ 0
ith constraint $=$	$\leftrightarrow$	ith variable is unrestricted
jth variable ≥ 0	$\leftrightarrow$	jth constraint $\leq$
jth variable ≤ 0	$\leftrightarrow$	jth constraint $\geq$
jth variable is unrestricted	$\leftrightarrow$	jth constraint $=$

Use the above rules and write the dual of the following primal LP:

$$\max\ 8x_1 + 3x_2 - 2x_3$$
$$\text{s.t.}\ \ x_1 - 6x_2 + x_3 \geq 2$$
$$5x_1 + 7x_2 - 2x_3 = -4$$
$$2x_1 - 3x_2 + 3x_3 \leq 3$$
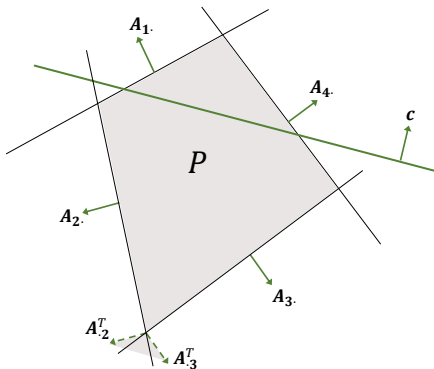$$x_1 \leq 0,\, x_2 \geq 0,\, x_3 \text{ unrestricted}$$

Duality

Notice that the primal variables are in $\mathbb{R}^n$ while the duals are in $\mathbb{R}^m$ so visualizing them in one place is difficult. However, consider a simple case where $n = 2, m = 4$ and the primals and duals are of the form

Primal Problem

$$\min \ \mathbf{c}^\mathsf{T}\mathbf{x}$$
$$\text{s.t. } \mathbf{Ax} \leq \mathbf{b}$$

Dual Problem

$$\max \ \mathbf{b}^\mathsf{T}\mathbf{y}$$
$$\text{s.t. } \mathbf{A}^\mathsf{T}\mathbf{y} = \mathbf{c}$$
$$\mathbf{y} \geq 0$$

Duality

Dual variables can also be interpreted in a third way. Consider the following LP that was discussed earlier.

$$
\begin{aligned}
\min \quad & -6x_1 - 4x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 = 6 \\
& 2x_1 + x_2 + x_4 = 9 \\
& 2x_1 + 3x_2 + x_5 = 16
\end{aligned}
$$

	x_1	x_2	x_3	x_4	x_5	RHS
RC	0	0	2	2	0	30
x_2	0	1	2	-1	0	3
x_1	1	0	-1	1	0	3
x_5	0	0	-4	1	1	1

What happens to the objective when we increase x_3 and x_4 by a small amount? This is same as decreasing the RHS of active constraints (6 and 9).

Duality
Another Interpretation

What are the reduced costs of the slack variables?

$$\bar{c}_3 = c_3 - \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{A}_{\cdot 3} = -\mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{A}_{\cdot 3}$$

It can be shown that y_1^* is equal to the negative of the reduced cost of the slack variable associated with the first constraint, i.e.,

$$y_1^* = -\bar{c}_3 = \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{A}_{\cdot 3}$$

In general, if the optimal primal is non-degenerate, the optimal dual variables are also called *marginal costs* and are equal to *the negative of the reduced costs of the slack variables.* In matrix notation,

$$\mathbf{y}^{*\mathsf{T}} = \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}$$

Duality

Consider the following primal and dual problems:

Primal LP	**Dual LP**
min $\mathbf{c}^\mathsf{T}\mathbf{x}$	max $\mathbf{b}^\mathsf{T}\mathbf{y}$
s.t. $\mathbf{A}\mathbf{x} \geq \mathbf{b}$	s.t. $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \geq \mathbf{0}$

Theorem (Weak Duality Theorem)

If $\mathbf{x}$ is feasible to the primal and $\mathbf{y}$ is feasible to the dual, then $\mathbf{c}^\mathsf{T}\mathbf{x} \geq \mathbf{b}^\mathsf{T}\mathbf{y}$

Since $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ and $\mathbf{y} \geq \mathbf{0}$,

$$\mathbf{y}^\mathsf{T}\mathbf{A}\mathbf{x} \geq \mathbf{b}^\mathsf{T}\mathbf{y}$$

Similarly, $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$ and $\mathbf{x} \geq \mathbf{0}$ implies,

$$\mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}^\mathsf{T}\mathbf{x}$$

Duality

The weak duality theorem states that the primal and dual LPs approach their optima from the right and left respectively, but can we have a gap?

Theorem (Strong Duality Theorem)

If the primal and dual problems are feasible, then there exists $\mathbf{x}^*$ and $\mathbf{y}^*$ such that $\mathbf{c}^\mathsf{T}\mathbf{x}^* = \mathbf{b}^\mathsf{T}\mathbf{y}^*$

(Sketch.) The optimum duals are the negative of the reduced costs of the slack variables. Hence, $\mathbf{y}^{*\mathsf{T}} = \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}$ and $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$.

$$\mathbf{c}^\mathsf{T}\mathbf{x}^* = \mathbf{c}_B^\mathsf{T}\mathbf{x}_B = \mathbf{c}_B^\mathsf{T}\mathbf{B}^{-1}\mathbf{b} = \mathbf{y}^{*\mathsf{T}}\mathbf{b}$$

Thus, one can solve the primal or dual and get the same optimum objective!

Notice that in both theorems, we assumed feasible solutions to the primal and dual problems. In general, we can have the following scenarios

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	✓	×	×
Unbounded	×	×	✓
Infeasible	×	✓	✓

Duality

Theorem

The pair $(\mathbf{x}^*, \mathbf{y}^*)$ are optimal to the primal and dual LPs $\Leftrightarrow$

$$y_i^*(\mathbf{A}_i.\mathbf{x}^* - b_i) = 0 \,\forall, i = 1, \ldots, m$$
$$(c_j - \mathbf{y}^{*\mathsf{T}}\mathbf{A}_{.j})x_j^* = 0 \,\forall j = 1, \ldots, n$$

Suppose we have optimality of $(\mathbf{x}^*, \mathbf{y}^*)$ pair. Then, according to strong duality and weak duality

$$\mathbf{c}^\mathsf{T}\mathbf{x}^* = \mathbf{y}^{*\mathsf{T}}\mathbf{A}\mathbf{x}^* = \mathbf{b}^\mathsf{T}\mathbf{y}^*$$

Hence $\sum_{i=1}^{m} y_i^*(\mathbf{A}_i.\mathbf{x}^* - b_i) = 0$ and $\sum_{j=1}^{n}(c_j - \mathbf{y}^{*\mathsf{T}}\mathbf{A}_{.j})x_j^* = 0$. Feasibility implies the complementary slackness conditions.

Duality

Consider the following LP that was discussed earlier.

$$
\begin{aligned}
\min \quad & -6x_1 - 4x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 = 6 \\
& 2x_1 + x_2 + x_4 = 9 \\
& 2x_1 + 3x_2 + x_5 = 16
\end{aligned}
$$

	x_1	x_2	x_3	x_4	x_5	RHS
RC	0	0	2	2	0	30
x_2	0	1	2	-1	0	3
x_1	1	0	-1	1	0	3
x_5	0	0	-4	1	1	1

Can you find the optimal dual solutions using the complementary slackness conditions?

Duality

Recall that the dual variables/marginal costs satisfy the following equation

$$\mathbf{y}^\mathsf{T} = \mathbf{c}_B^\mathsf{T} \mathbf{B}^{-1}$$

Let us take the primal problem in standard form.

Primal LP	**Dual LP**
$\min \ \mathbf{c}^\mathsf{T}\mathbf{x}$	$\max \ \mathbf{b}^\mathsf{T}\mathbf{y}$
s.t. $\mathbf{Ax} = \mathbf{b}$	s.t. $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}$	

What does the reduced cost optimality condition look like in terms of the dual variables? Having non-negative reduced costs is equivalent to achieving dual feasibility!

The primal simplex method tries maintains primal feasibility and attempts to achieve dual feasibility.

In the dual simplex method, we maintain dual feasibility and try to achieve primal feasibility.

We start with an initial dual basic feasible solution $\mathbf{y}^\mathsf{T} = \mathbf{c}_B^\mathsf{T} \mathbf{B}^{-1}$. This is easy when $\mathbf{c} \geq 0$ and the primal constraints are of the form $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ or $\mathbf{A}\mathbf{x} \leq \mathbf{b}$. Why?

Duality

Consider the following example for the purpose of illustration

$$
\begin{aligned}
\min \ & x_1 + x_2 \\
\text{s.t. } & x_1 + 2x_2 \geq 2 \\
& x_1 \geq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Using surplus variables, the standard form can be written as

$$
\begin{aligned}
\min \ & x_1 + x_2 \\
\text{s.t. } & x_1 + 2x_2 - x_3 = 2 \\
& x_1 - x_4 = 1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Note that in the dual simplex we do not write the dual and create a simplex tableau. We still work with the original problem.

The tableau is created as before and we begin by ensuring that the reduced costs above the basic variables are zeros. However, the basic variables *need not be feasible to the primal*.

	x_1	x_2	x_3	x_4	RHS
RC	1	1	0	0	0
x_3	-1	-2	1	0	-2
x_4	-1	0	0	1	-1

Identify a leaving variable (*pivot row*) using the most negative basic variable. This helps move towards primal feasibility.

Duality

Identify an entering variable using the min-ratio test. Select non-basic columns in the pivot row. Identify the negative elements in the row and divide the RC with the *absolute* values of the corresponding terms.

	x_1	x_2	x_3	x_4	RHS
RC	1	1	0	0	0
x_3	-1	-2	1	0	-2
x_4	-1	0	0	1	-1
min-ratio	1	1/2	-	-	

The min-ratio rule ensures that the reduced costs remain non-negative, thus dual feasibility is preserved. This identifies the *pivot column* and *pivot element*.

Duality

Carry out row operations to produce an identity matrix and zero reduced costs under the new basis.
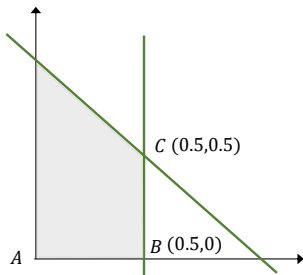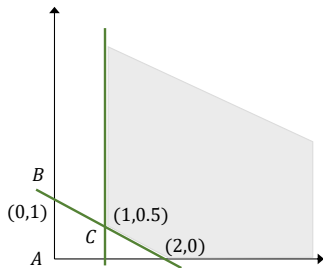
$$R1 \leftarrow R1/2$$
$$R0 \leftarrow R0 - R1$$

	x_1	x_2	x_3	x_4	RHS
RC	1/2	0	1/2	0	−1
x_2	1/2	1	−1/2	0	1
x_4	−1	0	0	1	−1

The min-ratio rule ensures that the reduced costs remain non-negative, thus dual feasibility is preserved. This identifies the *pivot column* and *pivot element*.

Duality

	x_1	x_2	x_3	x_4	RHS
RC	0	0	1/2	1/2	$-3/2$
x_2	0	1	$-1/2$	1/2	1/2
x_1	1	0	0	-1	1

Lecture Outline

Sensitivity Analysis

Sensitivity Analysis

Introduction

The parameters in an LP, i.e., $\mathbf{A}, \mathbf{b}$, and $\mathbf{c}$ are assumed to be given and known.

But decision makers may want to know what happens if the input data changes. Specifically, the following questions are relevant in this context.

1. If the problem data is different, is the current solution still optimum?

2. What is the range of values a variable can take for the current solution be optimum?

3. If the current solution is not optimum, should we start simplex from scratch or can we recycle?

For combinatorial optimization problems, two types of sensitivity studies are important. What happens if a new constraint and new variable is added.

Sensitivity Analysis

This study is also relevant when LPs serve as sub problems to other optimization problems where it is solved repeatedly with different input data.

The above questions will be analyzed by changing one parameter at a time. Imagine partial derivatives. If more than one variable changes, a similar approach can be followed.

Consider the following example for motivating sensitivity analysis.

$$\min \ -20x_1 - 25x_2 + 5x_3 - 30x_4$$
$$\text{s.t.} \ 3x_1 + 3x_2 + x_3 + 3x_4 \leq 45$$
$$x_1 + 2x_2 - 3x_3 + 3x_4 \leq 30$$
$$\mathbf{x} \geq \mathbf{0}$$

Suppose x_3 and x_4 are basic in the optimal solution. Can you find the optimal $\mathbf{x}$ values and the objective. Also calculate the reduced costs and the optimal dual solutions.

Sensitivity Analysis

What happens if the cost coefficient of the basic and non-basic variables change?

If the basis is unchanged, change to the cost will not change $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$. Thus, we just need to check if the reduced costs are greater than or equal to zero.

▶ Find the range of values for which the cost coefficient of a non-basic variable does not affect the basis.

▶ Find the range of values for which the cost coefficient of a basic variable does not affect the basis.

If the cost coefficient changes to a value beyond these limits, should we restart simplex?

Sensitivity Analysis

What happens when the b vector is modified?

Changes to the RHS does not affect reduced cost calculations but can affect feasibility. Hence we just need to check if $\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}$ is still greater than or equal to zero, i.e., it is basic feasible.

If $\mathbf{B}^{-1}\mathbf{b}$ is not basic feasible, do we have to restart simplex? Yes, if we are using the primal simplex. However, it is easier to dual simplex in this case since reduced costs are greater than or equal to zero at the current solution.

Sensitivity Analysis

What happens when a new constraint is added? If the current solution satisfies the new constraint, it is optimal to the original problem.

Else, one could use dual simplex instead of restarting primal simplex. The slack variable from the new constraint can be used as a new column in the basis and hence the reduced costs remain non-negative.

What happens when a new variable is added? Say a new decision variable x_7 is introduced in the example problem with cost coefficient -20 and $\mathbf{A}_{.7}^{\mathsf{T}} = \begin{bmatrix} 3 & 4 \end{bmatrix}$

This variable can be assumed to be non-basic and the current solution is basic feasible to the new problem. We hence need to check if the new variable can enter the basis.

Lecture Outline

CPLEX

CPLEX

CPLEX is a solver for linear and integer programs. See this link for the documentation. Explore the following tasks with the diet problem.

1. Give names to variables and constraints which will make it easy to debug your code. Check them using functions within model.variables and model.constraints.

2. Solve the problem using the primal and dual simplex methods. Use model.parameters to set the solution algorithm.

3. Construct the dual of the original problem and solve it using the primal method.

4. Find the basis, basis inverse, slacks, reduced costs, sensitivity ranges, and the indices of variables which belong to the basis after solving the problem. Use functions in model.solution, model.solution.advanced, and model.solution.sensitivity.

CPLEX

1. Export the problem to a .lp and a .sol file

2. Try solving the problem with a subset of the original constraints. Resolve by adding the left out constraints using both primal and dual simplex. Observe the difference in run times.

3. Run CPLEX by initializing a basic feasible solution that is different from the default one created using the slack variables.

4. Print the iterations of simplex using SimplexCallback. See lpex4.py for more details.

Your Moment of Zen