CE 205A
Transportation Logistics

Lecture 18

Collaborative Logistics

Previously on Transportation Logistics

The two-index formulation keeps track of binary variables x_{ij} which is 1 if arc (i, j) is used and is 0 otherwise.

$$
\begin{aligned}
\min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij} & \\
\text{s.t.} \quad & x(\delta^+(0)) = x(\delta^-(0)) = |K| & \\
& x(\delta^+(i)) = 1 & \forall\, i \in N \backslash \{0\} \\
& x(\delta^-(i)) = 1 & \forall\, i \in N \backslash \{0\} \\
& x(\delta^+(S)) \geq r(S) & \forall\, S \subseteq N \backslash \{0\}, S \neq \emptyset \\
& x_{ij} \in \{0, 1\} & \forall\, (i, j) \in A
\end{aligned}
$$

The fourth constraint, also called as the *capacity-cut constraints* (CCC) addresses both capacity limits as well as prevents sub-tours.

Previously on Transportation Logistics

Suppose J is a set of tours which are feasible (satisfy capacity constraints). Let a_{ij} be 1 if tour j visits customer i and is 0 otherwise.

Define c_j as the cost of the tour and y_j as a binary variable which is 1 if tour j is chosen.

$$
\begin{aligned}
\min \quad & \sum_{j \in J} c_j y_j \\
\text{s.t.} \quad & \sum_{j \in J} y_j = |K| \\
& \sum_{j \in J} a_{ij} y_j = 1 && \forall\, i \in V \setminus \{0\} \\
& y_j \in \{0, 1\} && \forall\, j \in J
\end{aligned}
$$

Lecture Outline

Lecture Outline

TSP and VRP Games

TSP and VRP Games

Introduction

Suppose a traveler is sponsored by the places they visit (imagine a speaker going to different events to give talks). What is a fair way to distribute the cost of the tour among the sponsors?

In the VRP context, how much should each customer pay for the set of routes that the trucks take.

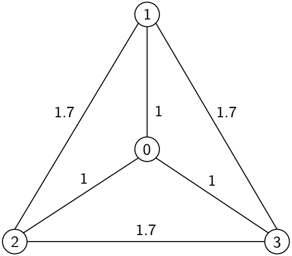
These are examples of co-operative games where the players are the cities or sponsors in the case of TSP and customers in the VRP case.

Cost allocations must be designed such that subsets of players do not withdraw and form smaller coalitions.

TSP and VRP Games

Example

Suppose each customer has a unit demand and we have three vehicles with capacity two. What is the optimal VRP solution? How should we divide the cost?



What if customer 1 has two units of demand?

Lecture Outline

Core

Core
Cost Allocations

Suppose y_i is the cost assigned to customer $i \in N$. What are some desirable conditions that ys must satisfy to make the allocation fair?

Let $c(S)$ be the optimal VRP solution for serving customers in a subset $S \subset N$. This is also called the *characteristic function*.

Then, the following conditions should be acceptable to the customers and no one has an incentive to deviate.

$$\sum_{i \in N} y_i = c(N) \qquad \text{Efficiency}$$

$$\sum_{i \in S} y_i \leq c(S) \ \forall \ S \subset N \qquad \text{Core-Defining Inequality (CDI)}$$

The CDI condition captures *individual rationality* when S is a singleton.

Properties of the characteristic function:

- Monotonicity: $c(S) \leq c(T)$ if $S \subset T \subset N$
- Subadditivity: $c(S) + c(T) \geq c(S \cup T)$, $\forall\ S, T \subset N, S \cap T \neq \emptyset$

The set of all ys that satisfy the previously defined conditions are called the *Core* and denoted as C. Show that each $y \in C$ is non-negative.

How many CDIs are needed to define C? Can we reduce them?

Suppose $\mathcal{S}$ is the set of all customers who can be serviced by a single truck, i.e., $\sum_{i \in S} d_i \leq C$, where $S \in \mathcal{S}$. We will call this a *feasible coalition*.

Theorem

CDIs defined for $S' \notin \mathcal{S}$ and $S' \subset N$ are redundant.

Proof.

(WTS) $\sum_{i \in S'} y_i \leq c(S')$, given $\sum_{i \in S} y_i \leq c(S)\ \forall\ S \in \mathcal{S}$

Suppose the optimal VRP partition for S' is $\{S_1, S_2, \ldots, S_m\}$

$$\sum_{j=1}^{m} \sum_{i \in S_j} y_i = \sum_{i \in S'} y_i$$

$$\sum_{j=1}^{m} c(S_j) = c(S')$$

Adding CDIs for feasible coalitions, $\sum_{j=1}^{m} \sum_{i \in S_j} y_i \leq \sum_{j=1}^{m} c(S_j)$, it follows that the CDI for S' is redundant. ■

Core

Key Results

Theorem

Consider an optimal VRP partition $S_1, \ldots, S_m$. Then

$$\sum_{i \in S_j} y_i = c(S_j) \; \forall \; y \in C, j = 1, \ldots, m$$

Proof.

$$c(N) = \sum_{i \in N} y_i = \sum_{j=1}^{m} \sum_{i \in S_j} y_i \leq \sum_{j=1}^{m} c(S_j) = c(N)$$

∎

Core

Key Results

Theorem

The core of a VRP game is non-empty iff its LP relaxation objective equals the integral solution $z_{LP} = z_{IP}$.

Proof.

The core is non-empty iff the following problem has an objective $z' \geq c(N)$.

$$z' = \max \sum_{i \in N} y_i$$

$$\text{s.t.} \sum_{i \in S} y_i \leq c(S) \; \forall \; S \in \mathcal{S}$$

Write the dual of the above problem. $z' = z_{LP} \leq z_{IP}$. Therefore, $z' \geq c(N) \Leftrightarrow z_{LP} = z_{IP}$. ∎

In summary, given an optimal VRP partition of N as $S_1, \ldots, S_m$, the core comprises of cost allocations y that satisfy

$$C = \left\{ y \;\middle|\; \begin{aligned} &\sum_{i \in S_j} y_i = c(S_j) \; \forall \; j \in \{S_1, \ldots, S_m\} \\ &\sum_{i \in S} y_i \leq c(S), S \in \mathcal{S} \setminus \{S_1, \ldots, S_m\} \\ &\sum_{i \in N} y_i = c(N) \end{aligned} \right\}$$

This set may contain uncountable number of cost-allocations. So, which one should we pick?

Lecture Outline

Nucleolus

Given a cost allocation vector y, the *excess* of S is defined as $c(S) - \sum_{i \in S} y_i$.

Coalitions with the smallest excess are the most disappointed. Hence a cost allocation y^1 is better than y^2 if

$$\min_{S \in \mathcal{S}} \left\{ c(S) - \sum_{i \in S} y_i^1 \right\} > \min_{S \in \mathcal{S}} \left\{ c(S) - \sum_{i \in S} y_i^2 \right\}$$

Suppose we sort the excess vector in $\mathbb{R}^{|\mathcal{S}|}$ in increasing order. Consider cost allocations that satisfy the efficiency and individual rationality conditions.

$$Y = \left\{ y \mid \sum_{i \in N} y_i = c(N); y_i \leq c(\{i\}) \right\}$$

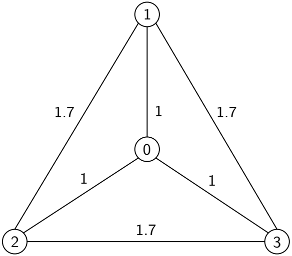
Nucleolus

Excess

The goal could then be to find the cost allocation that maximizes the minimum excess among all feasible coalitions.

The *nucleolus* is a cost allocation vector $y \in Y$ that has the greatest lexicographically ordered excess vector. E.g., $[1, 3, 5, 6] \succ [1, 3, 4, 8]$.

Calculate the nucleolus in the previous example.



Nucleolus

Row Generation

The problem of finding the nucleolus can be cast as

$$\max \min_{S \in \mathcal{S}} \left\{ c(S) - \sum_{i \in S} y_i \right\}$$

$$\text{s.t. } y \in Y$$

What kind of an optimization problem is this? Converting it into an LP

$$\begin{aligned} &\max w \\ \text{s.t. } &w \le c(S) - \sum_{i \in S} y_i \ \forall\ S \in \mathcal{S} \\ &y \in Y \end{aligned}$$

The number of constraints is exponential and hence we can solve this problem using a row generation approach.

Nucleolus

Row Generation

Let $\Omega \subseteq \mathcal{S}$ be a subset of feasible coalitions. The restricted master problem (RMP) can be written as

$$\begin{aligned} & \max w \\ \text{s.t.} \ & w \leq c(S) - \sum_{i \in S} y_i \ \forall \ S \in \Omega \\ & y \in Y \end{aligned}$$

Based on the solution y^*, w^*, we can find if the left-out constraints are violated by solving a new optimization sub-problem

$$\begin{aligned} & \min -w^* + c(S) - \sum_{i \in S} y_i^* \\ \text{s.t.} \ & S \in \mathcal{S} \setminus \Omega \end{aligned}$$

When do we terminate the RMP? What kind of an optimization problem is this?

Nucleolus

Row Generation

Define a vector s such that $s_i = 1$ if i in coalition S and is 0 otherwise. Using this definition, we can rewrite the constraint $S \in \mathcal{S} \setminus \Omega$ as (why?)

$$\sum_{\{i:s_i^j=0\}} s_i + \sum_{\{i:s_i^j=1\}} (1 - s_i) \geq 1 \; \forall \; j \text{ such that } S_j \in \Omega$$

Therefore, the sub-problem can be rewritten as

$$\begin{aligned} & \min -w^* + c(s_1, \ldots, s_n) - \sum_{i \in N} y_i^* s_i \\ \text{s.t.} \quad & \sum_{\{i:s_i^j=0\}} s_i + \sum_{\{i:s_i^j=1\}} (1 - s_i) \geq 1 \; \forall \; j \text{ such that } S_j \in \Omega \\ & y \in Y \end{aligned}$$

This problem resembles the prize-collecting TSP with some additional constraints. Suppose we found a solution to the max-min nucleolus problem, can we terminate?

What would happen if we did not have the extra constraints?

Nucleolus

Lexicographic Ordering

If multiple cost allocations lead to the same w^*, ties are broken using the next highest excess.

E.g., consider three cost allocations for which the excess vectors are

$$[1, 3, 5, 6], [1, 3, 4, 8], [1, 3, 7, 7]$$

The coalitions corresponding to the green excess values may be same or different. Let $\Gamma = \{S_1, S_2, S_3\}$.

Maximize the second lowest excess using a new optimization problem.

$$\begin{aligned} & \min w_2 \\ \text{s.t. } & w_2 \leq c(S) - \sum_{i \in S} y_i \ \forall \ S \in \mathcal{S} \setminus \Gamma \\ & w^* = c(S) - \sum_{i \in S} y_i \ \forall \ S \in \Gamma \\ & y \in Y \end{aligned}$$

Nucleolus

Lexicographic Ordering

This process can be repeated by modifying these optimization problems until a unique solution is found.

Since the number of constraints are again exponential, these problems can also be solved using a row generation procedure.

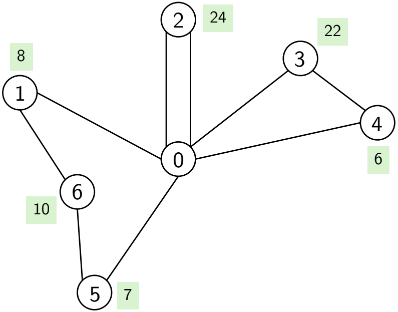
In finding the nucleolus, why do we care about all feasible coalitions and not just the ones in the optimal VRP partition?

Nucleolus

Example

Consider the optimal solution shown below for a scenario with three vehicles each with capacity of 30.

$$\begin{bmatrix} 0 & 24 & 19 & 20 & 27 & 16 & 12 \\ & 0 & 17 & 31 & 44 & 36 & 23 \\ & & 0 & 16 & 29 & 35 & 25 \\ & & & 0 & 15 & 34 & 28 \\ & & & & 0 & 40 & 37 \\ & & & & & 0 & 13 \\ & & & & & & 0 \end{bmatrix}$$



Write down the constraints for finding the core and the nucleolus.

Nucleolus

Example

{1}	48	{1,3}	75	{1,5,6}	76
{2}	38	{1,4}	96	{1,4,5}	123
{3}	40	{1,5}	76	{1,4,6}	106
{4}	54	{1,6}	59	{4,5,6}	92
{5}	32	{2,4}	75		
{6}	24	{3,4}	62		
		{3,5}	70		
		{4,5}	83		
		{4,6}	76		
		{5,6}	41		

From the optimal solution, some of the constraints would appear with equality conditions based on the earlier proposition. That is,

$$y_2 = 38 \qquad y_3 + y_4 = 62 \qquad y_1 + y_5 + y_6 = 76$$

If the core is empty, these would remain as $\leq$ constraints.

References

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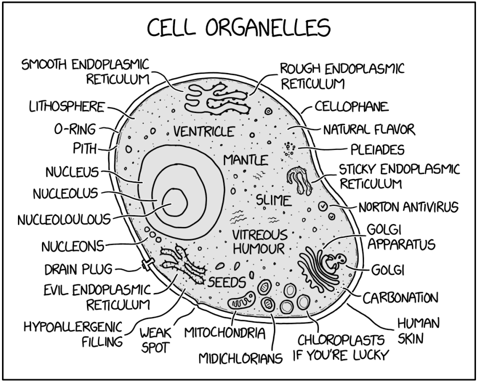
Your Moment of Zen



Source: xkcd