CE 205A
Transportation Logistics

Lecture 13

Cuts for TSPs – Part I

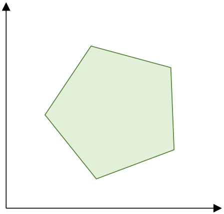
Previously on Transportation Logistics

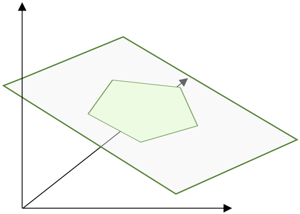
Let $X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A} \leq \mathbf{b}\}$. Although, we use a $\leq$ sign, assume that some or all of the rows can have an equality sign.

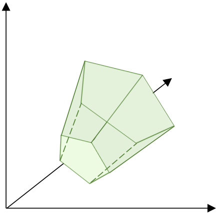
Definition (Dimension)

A polyhedron X is of dimension k, denoted by, $\dim(X) = k$, if the maximum number of affinely independent points in X is $k + 1$.

What are the dimensions of the following polyhedra?

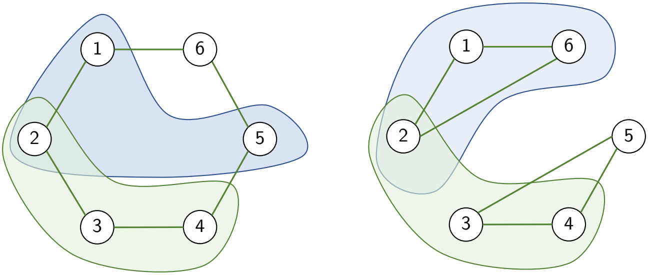






Previously on Transportation Logistics

What if we did not have sub-tour elimination constraints? When is the constraint strictly $<$ and when is it $=$?



An alternate way of writing the sub-tour elimination constraints (SEC) is

$$\sum_{u \in S} \sum_{v \in S^c} x_{\{u,v\}} \geq 2 \qquad \forall\, S \subset V, S \neq \emptyset$$

Apply this version to the above examples? Show that the two SEC are equivalent.

Previously on Transportation Logistics

The following short-hand notation is widely used in TSP literature. Let $S \subseteq V$.

- $E(S)$: Edges with both end points in S (also called the *edge set*).
- $\delta(S)$ or $\delta(S, S^c)$: Set of edges with one end in S and another in S^c (also called the *cut set*).
- If $|S| = 1$, we write $\delta(u)$ instead of $\delta(\{u\})$ to indicate the set of edges which have u as one end point.

Additionally, we define $x(E(S))$ and $x(E(S))$ as follows

$$x(E(S)) = \sum_{u \in S} \sum_{v \in S} x_{\{u,v\}}$$

$$x(\delta(S)) = \sum_{u \in S} \sum_{v \in S^c} x_{\{u,v\}}$$

The xs are also called *incidence vectors*. The subgraph with edges for which x values are positive is called the *support graph*.

Previously on Transportation Logistics

The DFJ formulation using E-δ notation can be written as

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & x(\delta(u)) = 2 && \forall\, u \in V \\ & x(E(S)) \leq |S| - 1 && \forall\, S \subset V, S \neq \emptyset \\ & x_e \in \{0, 1\} && \forall\, e \in E \end{aligned}$$

The formulation with the alternate SEC constraints take the form

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & x(\delta(u)) = 2 && \forall\, u \in V \\ & x(\delta(S)) \geq 2 && \forall\, S \subset V, S \neq \emptyset \\ & x_e \in \{0, 1\} && \forall\, e \in E \end{aligned}$$

We can further restrict $3 \leq |S| \leq |V|/2$. (Why?)

Lecture Outline

1. Connectivity Cuts
2. Blossoms and Combs

Lecture Outline

Connectivity Cuts

Cutting Planes

TSP Cuts

Dantzig et al.'s 49-city tour paper was one of the first to suggest the cutting plane method for TSPs. Instead of the exponential number of sub-tour elimination constraints, they add them every time such constraints are violated.

The end result is a graph that is fully connected but some of the flows can be still non-integral. A new family of cuts are generated using more involved graph theoretical concepts to get to the final optimal solution.

Connectivity Cuts

Introduction

The DFJ formulation has an exponential number of constraints of which only a subset are tight at the optimal solution.

This motivates the idea of using row generation or lazy cuts (within a branch and cut scheme) for solving the TSP.

Suppose we solve the LP relaxation of the TSP problem with just degree constraints. The resulting solution may be fractional or may not be a tour. Hence, the violated constraint can be identified and added iteratively.

Recall that the MTZ formulation has fewer constraints but the LP relaxation is not strong. In fact, SEC have the following property.

Theorem (Grötschel and Padberg (1979))

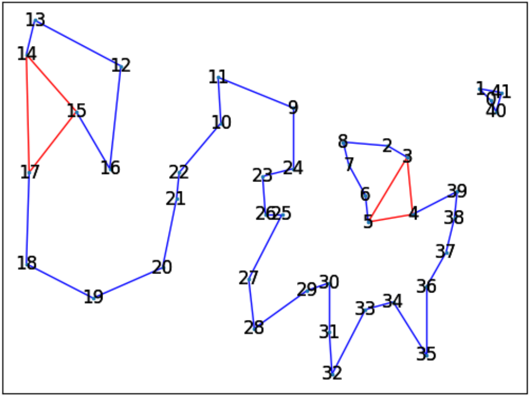
The subtour elimination constraints $x(\delta(S)) \geq 2$ *for all* $S \subset V$, $3 \leq |S| \leq |V|/2$ *are facets of the convex hull of feasible TSP tours.*

Connectivity Cuts

Introduction

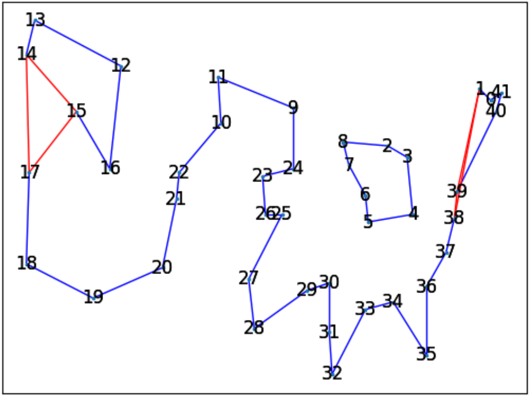
Consider Dantzig's 42-city network. The LP solution lower bound without SECs is 641 (compared to the optimal IP solution of 699) and has fractional variables and is not a tour because of the disconnected components.

Original LP = 641.0

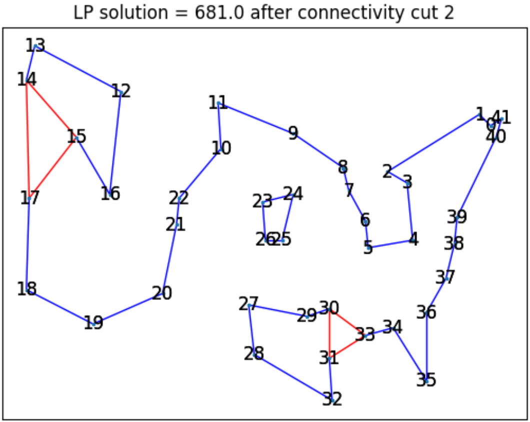


After adding the SEC with $S = \{0, 1, 40, 41\}$, the North-East part of the tour gets connected but a new subtour is created.

LP solution = 676.0 after connectivity cut 1



Add a new SEC with $S = \{2, 3, 4, \dots, 8\}$.

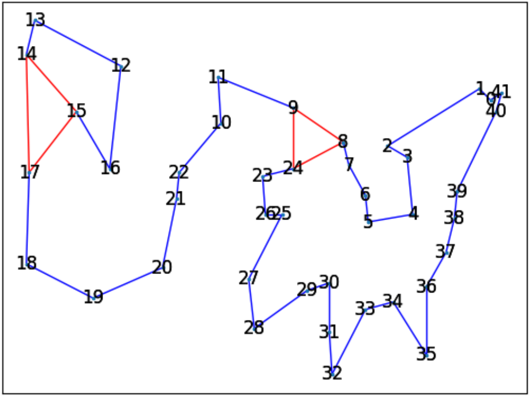


Connectivity Cuts

Subtour Elimination

Add a new SEC with $S = \{23, 24, 25, 26\}$.

LP solution = 682.5 after connectivity cut 3



The solution is connected, but is not integral or a tour. Are all SECs satisfied?

Connectivity Cuts

2-Connected Cuts

Definition (k-connected)

A graph with at least k vertices is said to be k-connected if it remains connected after removing fewer than k vertices.

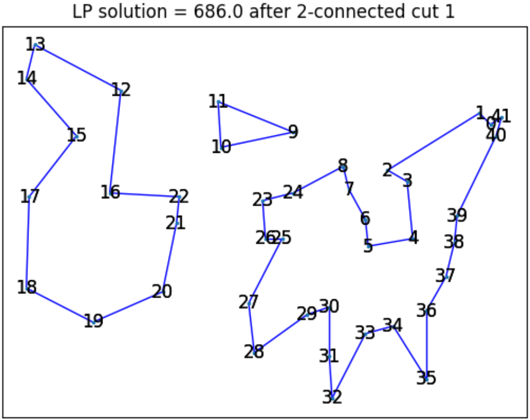
Are TSP tours k-connected? If so, for what values of k? Is the support graph from previous LP solution 2-connected?

Definition

An articulation point or cut vertex is a vertex when removed (along with the edges incident to it) disconnects the graph.

Can you identify articulation points in the previous support graph? To make the graph 2-connected, we can add SECs based on the connected components created after deleting the articulation points.

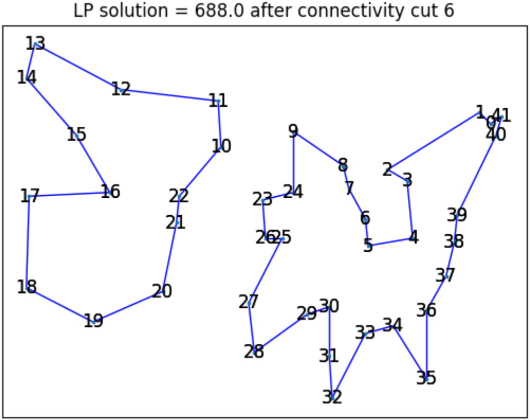
The support graph after adding 2-connected cuts using articulation point 17 has a better LP relaxation but is disconnected. Is this 2-connected? How should we proceed? We can add three connectivity cuts.



Connectivity Cuts

Looping Through SEC and 2-Connected Cuts

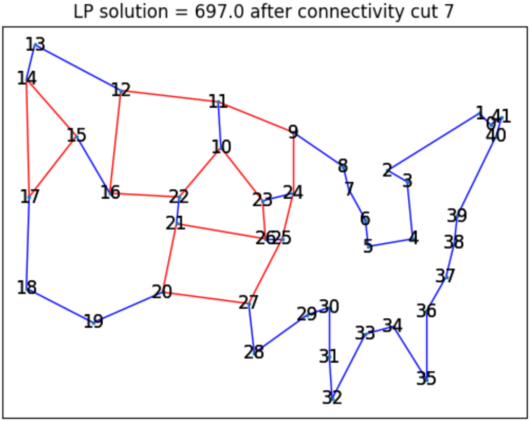
The new solution does not violate the added cuts. Add another connectivity cut.



Connectivity Cuts

Looping Through SEC and 2-Connected Cuts

This solution is connected and 2-connected. However, there are many edges with fractional LP solutions.



Connectivity Cuts

Looping Through SEC and 2-Connected Cuts

This solution satisfies the SECs? So why is it not optimal? Some of the variables are still fractional. This solution corresponds to solving

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e & \\ \text{s.t.} \quad & x(\delta(u)) = 2 & \forall\, u \in V \\ & x(\delta(S)) \geq 2 & \forall\, S \subset V, S \neq \emptyset \\ & x_e \in [0, 1] & \forall\, e \in E \end{aligned}$$

At this stage, one could branch on one of the fractional variables. This may lead to LP relaxations that violate SECs. Hence, connectivity cuts can be added as lazy cuts.

Theorem (Grötschel and Padberg (1979))

The bounds $0 \leq x_e \leq 1$ for all $e \in E$ also define facets of the convex hull of feasible TSP tours.

This method generalizes the previous two approaches on connectivity and 2-connectedness and can provide constraints that result in maximum violations of SECs.

The idea is to simply find the global min-cut between any s-t pair in the support graph, i.e., find $S \subset V$ such that $\min x^{\text{LP}}(\delta(S))$. If this is 2, we can generate a cut.

For instance, solving this would give 0 for disconnected graphs and an objective less than or equal to 1 for support graphs that are not 2-connected (e.g., $\{12, 13, 14, 15, 16\}$ in the instance with LP solution $= 682.5$).

Connectivity Cuts

Hong-Padberg's Separation Procedure

How many min-cuts should we solve? An obvious choice is to choose every pair of s and t vertices, i.e., $\binom{n}{2}$ min-cut problems. Can we do better?

Fix s and iterate across all ts. You can find the global min-cut by solving $n-1$ min-cut problems. (Why?)

Blossoms and Combs

Blossoms and Combs

Valid Inequalities

DFJ used other valid inequalities that helped them to find the optimal solution without having to branch. These valid inequalities were explored formally in the subsequent years.

Proposition

The dimension of the convex hull of feasible TSP tours is $\binom{n}{2} - n$*, where* n *is the number of vertices.*

Hence, the TSP polytope is not full-dimensional, which is expected because of the equality constraints. This makes it harder to prove that certain valid inequalities are facets and there are infinitely many of them.

Yet, several interesting families of facet-defining valid inequalities are known to exist.

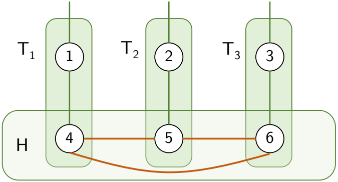
Blossoms and Combs

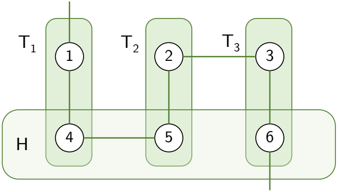
Definition

Consider a set of subsets $H, T_1, T_2, \ldots, T_k$ of V satisfying the following conditions.

- $|T_i| = 2$ and each T_i has a vertex in H and one in H^c.
- $T_1, T_2, \ldots, T_k$ are pairwise disjoint.
- $k \geq 3$ and is odd.

In this example, $T_1 = \{1, 4\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 6\}$, and $H = \{4, 5, 6\}$. The T sets are also called *teeth* and H is called the *handle*.





Blossoms and Combs

Blossom Inequalities

Subsets which satisfy these conditions are called blossoms and can be used to generate Blossom inequalities or 2-matching inequalities of the form

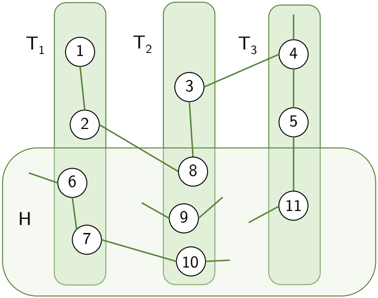
$$x\left(\delta(H)\right) + \sum_{i=1}^{k} x\left(\delta(T_i)\right) \geq 3k + 1$$

Using the degree constraints $x(\delta(S)) = 2|S| - 2x(E(S))\,\forall\, S \subseteq V$ (Why?). Hence, blossom inequalities can also be written as

$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leq |H| + \sum_{i=1}^{k} |T_i| - \frac{3k+1}{2}$$

The definition of Blossoms can be extended to combs which are subsets that satisfy the following conditions:

- Each T_i has at least one vertex in H and at least one in H^c.
- $T_1, T_2, \ldots, T_k$ are pairwise disjoint.
- $k \geq 3$ and is odd.

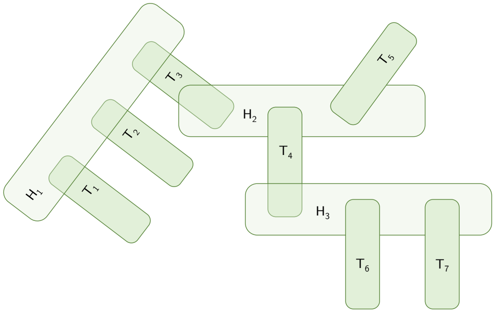


Thus, every blossom is a comb but not vice versa. The valid inequalities take the same form as described earlier.

Blossoms and Combs

Clique Trees

There are extensions of combs called clique trees that can be also be used to generate valid inequalities. These have multiple non-intersecting handles and teeth attached to each handle that are non-intersecting as well.



Proposition (Grötschel and Padberg ('79), Grötschel and Pulleyblank ('86))

Blossom, comb, and clique-tree inequalities are facet-defining for the convex hull of feasible TSP tours.

While these subsets provide good valid inequalities, how can we identify them using a separation routine?

Polynomial time complexity methods such as the Padberg-Rao exist for finding violated blossom inequalities.

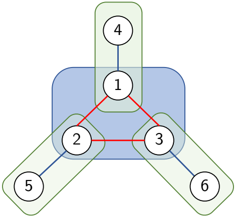
Most codes however rely on separation heuristics. For comb inequalities, they typically use network contraction methods (also called safe shrinking) to work with smaller support graphs.

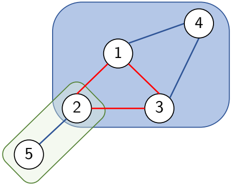
For example, consider a subset of the support graph for which edge weights are fractional. Call this $G_{1/2}$.

Blossoms and Combs

Separation Problems

Determine the set of connected components in $G_{1/2}$. Let the collection of these nodes be $S_1, S_2, \ldots, S_l$. Check if the set $T = \left\{e : e \in \delta(S_i), x_e^{\mathrm{LP}} = 1\right\}$ has odd cardinality. The following cases may arise:



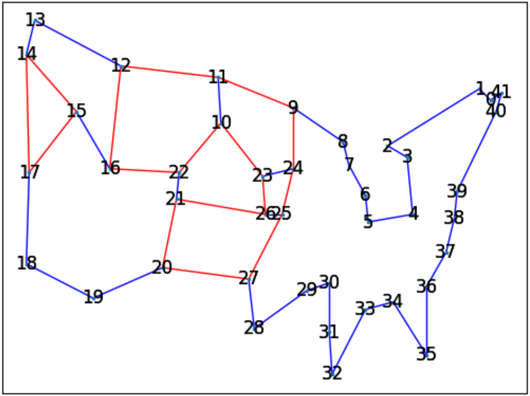


In the first case, edges of T are non-intersecting and hence they naturally form a blossom with $S_i = \{1, 2, 3\}$ as the handle.

In the second case, edges of T intersect outside S_i and hence we add the intersecting edge 4 to S_i. $S_i = \{1, 2, 3, 4\}$ provides an SEC.

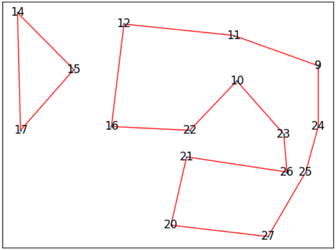
Can you spot a blossom or comb in inequality in the current solution?

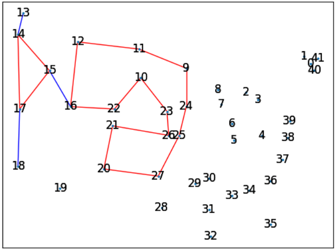
LP solution = 697.0 after connectivity cut 7



Blossoms and Combs

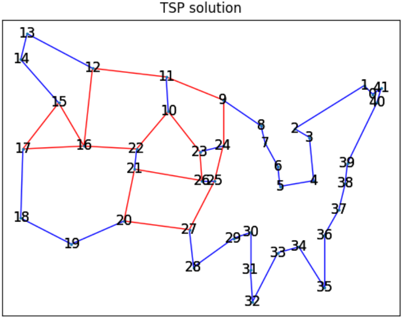
Example





Blossoms and Combs
Example

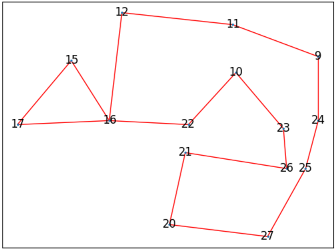
The LP relaxation solution increases to 698 after adding the blossom inequality.

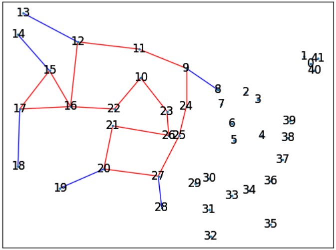


Blossoms and Combs

Example

The heuristic cannot be used in this case since there are even number of teeth.

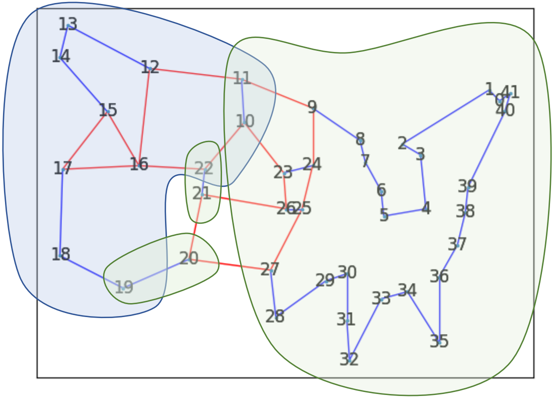




Blossoms and Combs

Example

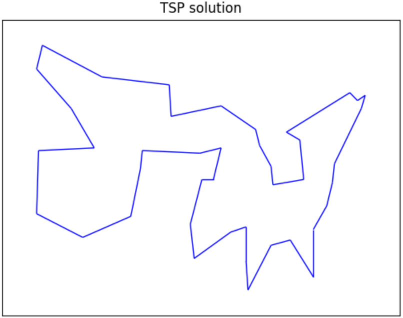
At this stage, one could search for comb inequalities or branch to find better solutions.



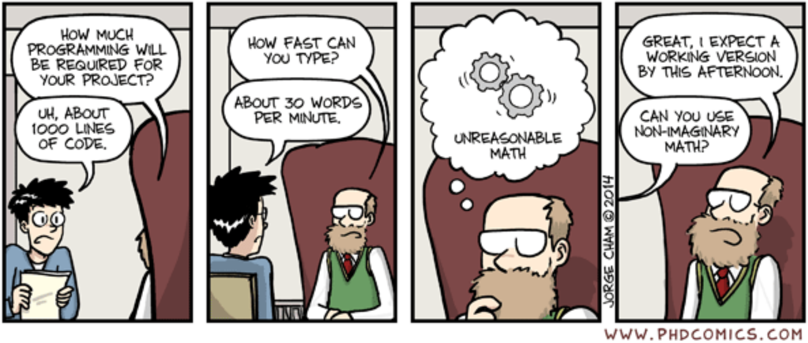
Blossoms and Combs

Example

The final optimal tour using the branch and cut has an objective of 698 and looks as shown below.



Your Moment of Zen



Source: xkcd