CE 205A
Transportation Logistics

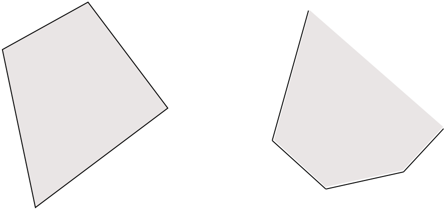
Lecture 10

Polyhedral Theory

Previously on Transportation Logistics

Definition (Polytope)

A polyhedron $P \subset \mathbb{R}^n$ is bounded, also called a polytope, if there exists a constant $C > 0$ such that $|x_i| \leq C \; \forall \, i = 1, \ldots, n$



All points inside a polytope can be expressed as a convex combination of its extreme points. Mathematically, let $X = \{\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be a polytope.

$$X = \left\{ \mathbf{x} : \mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}^i, \boldsymbol{\lambda} \geq \mathbf{0}, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

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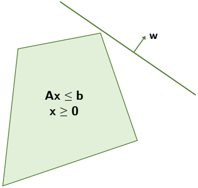
How do you check if a given inequality, e.g., $-11x_1 + 4x_2 \leq 6$ is valid?



Proposition

An inequality $\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0$ *is a valid inequality for* $X = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \Leftrightarrow \exists\ \mathbf{y} \geq \mathbf{0}$, *such that* $\mathbf{A}^\mathsf{T}\mathbf{y} \geq \mathbf{w}$ *and* $\mathbf{b}^\mathsf{T}\mathbf{y} \leq w_0$.

Proof.

$\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0$ is a valid inequality $\Leftrightarrow$

$$
\begin{aligned}
w_0 \geq \max\ & \mathbf{w}^\mathsf{T}\mathbf{x} \\
\text{s.t. } & \mathbf{Ax} \leq \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

The dual problem of the above LP is $\min \mathbf{b}^\mathsf{T}\mathbf{y}$ s.t., $\mathbf{A}^T\mathbf{y} \geq \mathbf{w}, \mathbf{y} \geq 0$. □

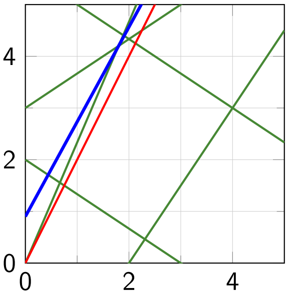
Previously on Transportation Logistics

Apply the CG procedure for the earlier example using $\boldsymbol{\lambda} = (7/30, 0, 0, 1/10, 0)$.

$$-\frac{49}{30}x_1 + \frac{7}{10}x_2 \leq 0$$
$$-\frac{2}{10}x_1 + \frac{3}{10}x_2 \leq \frac{9}{10}$$

Adding the above inequalities,

$$-\frac{55}{30}x_1 + x_2 \leq \frac{9}{10}$$



Rounding the LHS and RHS, we get $-2x_1 + x_2 \leq 0$, which is one of the inequalities describing the convex hull.

Previously on Transportation Logistics

Consider the Knapsack constraint $X = \{\mathbf{x} \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \leq b\}$. Let $N = \{1, \ldots, n\}$. Assume that $b > 0$ and $a_j > 0$ for all j. Is this restrictive?

Definition (Cover)

A set $C \subseteq N$ is a cover/dependent set if $\sum_{j \in C} a_j > b$. A cover is minimal if $C \backslash \{j\}$ is not a cover or any $j \in C$.

Determine all covers of $2x_1 + 5x_2 + 3x_3 + x_4 \leq 6$.

- Which of these are minimal?
- What kind of valid inequalities are implied by covers?

Proposition

If $C \subseteq N$ is a cover for X, then $\sum_{j \in C} x_j \leq |C| - 1$ is valid for X.

Previously on Transportation Logistics

Definition (Linear Independence)

A collection of vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^n$ is linearly independent if $\lambda_1 \mathbf{x}^1 + \lambda_2 \mathbf{x}^2 + \ldots + \lambda_k \mathbf{x}^k = \mathbf{0}$ implies that λ_is are zeros.

The rank of a matrix is the number of linearly independent rows or columns.

Definition (Span)

A collection of vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^n$ is said to span $\mathbb{R}^n$ if any vector $\mathbf{b} \in \mathbb{R}^n$ can be expressed as a linear combination of $(\mathbf{x}^1, \ldots, \mathbf{x}^k)$.

Definition (Basis)

A collection of vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^n$ is said to form a basis if it spans $\mathbb{R}^n$ and removing one vector results in a collection that does not span $\mathbb{R}^n$.

A collection of vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k \in \mathbb{R}^n$ forms a basis of $\mathbb{R}^n$ iff $k = n$ and the vectors are linearly independent.

Lecture Outline

1. Faces and Facets
2. Lifting Valid Inequalities

Faces and Facets

Faces and Facets

Background

Given a polytope X, we have methods to generate valid inequalities. But we saw that not all valid inequalities are useful.

In this context, the following questions are of interest.

1. Which constraints/valid inequalities of X are redundant. This gives us a "minimal description" of X.
2. More importantly, which valid inequalities of $\text{Conv}(X \cap \mathbb{Z}^n_+)$ "make up" the convex hull.

Faces and Facets

Background

Proposition (Dominance)

Let $X = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. Suppose $(\mathbf{w}, w_0)$ and $(\mathbf{v}, v_0)$ are valid for X. $\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0$ is said to dominate $\mathbf{v}^\mathsf{T}\mathbf{x} \leq v_0$ if $\exists\ \lambda > 0$ such that

$$\mathbf{w} \geq \lambda \mathbf{v}$$

$$w_0 \leq \lambda v_0$$

with at least one of the inequalities being strict.

In other words, every $\mathbf{x}$ that satisfies $\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0$ also satisfies $\mathbf{v}^\mathsf{T}\mathbf{x} \leq v_0$. Can we write $\Rightarrow$ in the definition?

Note that multiplying an inequality by a positive scalar will not change the inequality.

Faces and Facets

Background

Recall that non-negative linear combinations of valid inequalities generates another valid inequality. Any inequality that is dominated can be removed from the constraint set.

Definition (Redundance)

A valid inequality $(\mathbf{w}, w_0)$ is redundant if $\exists$ k valid inequalities $(\mathbf{w}^i, w_0^i)$ and weights $\lambda_i > 0$ for $i = 1, \dots, k$ such that

$$\sum_{i=1}^{k} \lambda_i \mathbf{w}^{i\mathsf{T}} \mathbf{x} \leq \sum_{i=1}^{k} \lambda_i w_0^i$$

dominates $\mathbf{w}^\mathsf{T} \mathbf{x} \leq w_0$.

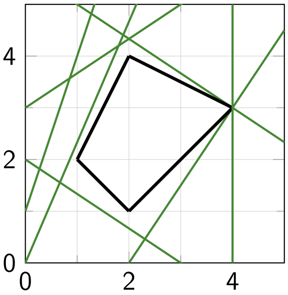
A valid inequality that is not dominated by other valid inequality is said to be *maximal*.

Faces and Facets

Background

Using the definitions, can you determine which of the following inequalities are redundant.

$$
\begin{aligned}
-7x_1 + 3x_2 &\leq 0 \\
-2x_1 - 3x_2 &\leq -6 \\
3x_1 - 2x_2 &\leq 6 \\
-2x_1 + 3x_2 &\leq 9 \\
-2x_1 - 3x_2 &\leq 17 \\
-3x_1 + x_2 &\leq 1 \\
x_1 &\leq 4 \\
x_1, x_2 &\geq 0
\end{aligned}
$$



$-3x_1 + x_2 \leq 1$ is dominated by $-7x_1 + 3x_2 \leq 0$. Choose $\lambda = 3$. Likewise, $x_1 \leq 4$ is redundant. (Why?)

Definition (Affine Independence)

A collection of vectors $\mathbf{x}^1, \dots, \mathbf{x}^k \in \mathbb{R}^n$ are affinely independent if the unique solution to $\sum_{i=1}^k \lambda_i \mathbf{x}^i = \mathbf{0}$, $\sum_{i=1}^k \lambda_i = 0$ is that all the λ_is are zeros.

In other words, a collection of vectors are affinely independent if no vector can be written as an affine combination of the other vectors.

The definition is equivalent to the following versions when $k \geq 2$.

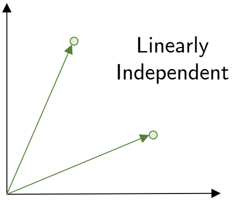
- $(\mathbf{x}^1, 1), \dots, (\mathbf{x}^k, 1)$ are linearly independent. (Why?)
- $\mathbf{x}^2 - \mathbf{x}^1, \dots, \mathbf{x}^k - \mathbf{x}^1$ are linearly independent. (Why?)

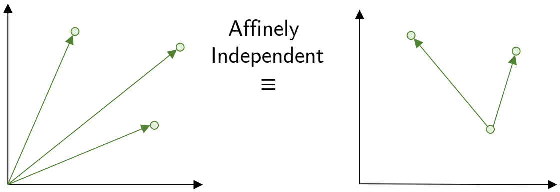
Affine independence allows us to mathematically describe the dimension of a polyhedra and identify hyperplanes that contribute to the convex hulls.

Faces and Facets

Affine Independence

Affine independence can be viewed as shifting the origin to one of the vectors and checking for linear independence.





Provide examples for the following scenarios in $\mathbb{R}^3$:

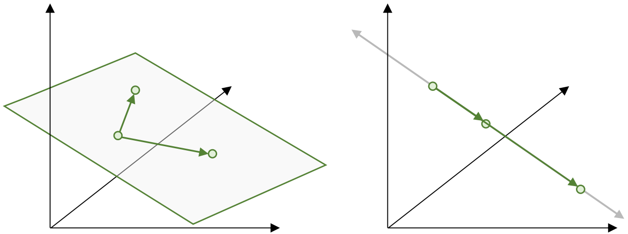
- Three linearly independent points.
- Four affinely independent points.
- Two vectors in that are linearly independent. Are they affinely independent?
- A collection of vectors that are affinely independent but not linearly independent.

Note that linear independence implies affine independence but not vice-versa.

Faces and Facets

Affine Independence

Three points in $\mathbb{R}^3$ are affinely independent if and only if there is a plane passing through them.



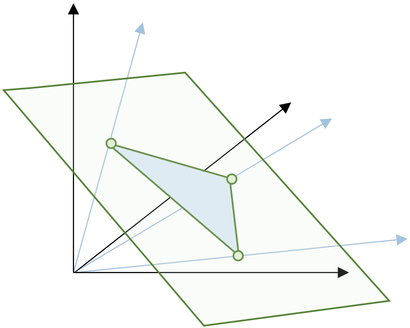
- What is the maximum number of vectors that can be linearly independent in $\mathbb{R}^n$? Affinely independent?
- Can the origin vector be a part of a collection of linearly independent vectors? Affinely independent vectors?

Faces and Facets

Affine Independence

Just line convex hulls, it is also possible to define affine hulls where the weights are not required to be non-negative. Suppose $X = \{x^1, \ldots, x^k\}$.

$$\text{aff}(X) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}^i : \lambda_i \in \mathbb{R}, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$



Faces and Facets

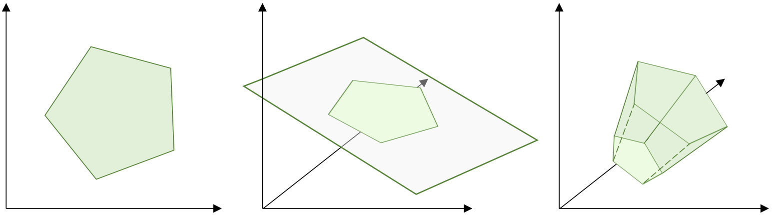
Dimension of Polyhedra

Let $X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A} \leq \mathbf{b}\}$. Although, we use a $\leq$ sign, assume that some or all of the rows can have an equality sign.

Definition (Dimension)

A polyhedron X is of dimension k, denoted by, $\dim(X) = k$, if the maximum number of affinely independent points in X is $k + 1$.

What are the dimensions of the following polyhedra?



Definition (Dimension)

X is *full dimensional* if $\dim(X) = n$.

Consider the node packing problem with feasible region given by

$$X = \{\mathbf{x} \in \{0,1\}^n : x_i + x_j \leq 1 \;\forall\; (i,j) \in E\}$$

What is $\dim(Conv(X))$? Select all unit vectors and the origin.

For the same reason, $\dim(Conv(X))$ of the knapsack polytope is n, where $X = \{\mathbf{x} \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b\}$. Note that we can safely assume $a_j \leq b$. (Why?)

Faces and Facets

Dimension of Polyhedra

Suppose we split the constraints into inequalities and equalities.

$$\begin{bmatrix} \mathbf{A}^{\leq} \\ \mathbf{A}^{=} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{b}^{\leq} \\ \mathbf{b}^{=} \end{bmatrix}$$

Proposition (Dimension)

$dim(X) + rank(\mathbf{A}^{=}, \mathbf{b}^{=}) = n.$

We say $\mathbf{x}$ is an interior point if $\mathbf{A}_{i.}\mathbf{x} < b_i$ for all $i = 1, \ldots, m$. In other words, no constraint is of the equality form. Thus a polyhedron is full dimensional iff it has an interior point.

What is $\dim(X)$ and $rank(\mathbf{A}^{=}, \mathbf{b}^{=})$ for the following set of constraints?

$$\begin{aligned} x_1 + x_3 &\leq 1 \\ x_1 + x_2 + 2x_3 &\leq 2 \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Faces and Facets

Face

Definition (Face)

Given a valid inequality $(\mathbf{w}, w_0)$ for X, a set of points $F = \{\mathbf{x} \in X : \mathbf{wx} = w_0\}$ is defined as the face of X.

We say the valid inequality $\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0$ *represents* or *defines* the face F.

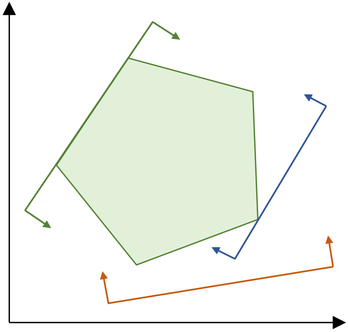
Can F be $\emptyset$ or X? If that is not the case, it said to be *proper*. If F is non-empty, we also say that $(\mathbf{w}, w_0)$ *supports* X.

To arrive at a minimal representation of a polyhedron, we can discard all valid inequalities/faces that do not support X.

Two faces F_1 and F_2 are unique only if $\text{aff}(F_1) \neq \text{aff}(F_2)$.

Definition (Facet)

A face F of X is a facet if $\dim(F) = \dim(X) - 1$



Which of the three inequalities are faces and facets? The blue valid inequality is similar to $x_1 \leq 4$ in the earlier example.

Faces and Facets

Facet

Proposition

If F is facet of X, then there exists some inequality $\mathbf{A}_{i\cdot}\mathbf{x} \leq b_i$ representing F.

Proposition

An face $\mathbf{A}_{i\cdot}\mathbf{x} \leq b_i$ which has dimension less than $dim(X) - 1$ is redundant.

Proposition

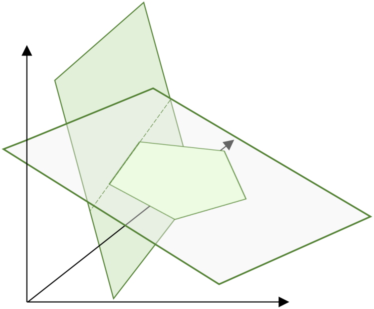
A full dimensional polyhedron X has a unique (allowing scalar multiplication) minimal representation by a finite set of linear inequality, each of which is a facet.

Faces and Facets

Facet

Proposition

If X is not full dimensional, i.e., $\dim(X) = n - k$, $k > 0$, then X can be described by k linearly independent rows of $\mathbf{A}^{=}, \mathbf{b}^{=}$ and a set of linear inequalities, each of which represents a facet.



Is this representation unique?

Faces and Facets

Example

Consider the polytope defined by the following constraints.

$$\begin{aligned} x_1 + x_3 &\leq 1 \\ x_1 + x_2 + 2x_3 &\leq 2 \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Determine if the following inequalities are valid. If they are, check if the faces defined by them are facets.

- $-x_1 - x_2 + x_3 \leq 1$
- $2x_1 - 7x_2 + 2x_3 \leq 2$

Faces and Facets

Useful Valid Inequalities

For solving integer programs, the ideal valid inequalities are those which define facets of $\text{Conv}(X)$. These cannot be dominated by other valid inequalities.

Suppose for a problem, you found a valid inequality. How can you check if it is facet inducing/defining?

- Show that $\dim(F)$ is $\dim(\text{Conv}(X)) - 1$. That is, show that there are $\dim(\text{Conv}(X))$ affinely independent points in F.
- However, determining $\dim(\text{Conv}(X))$ can be difficult. In such cases, we try to find facet-defining inequalities for the relaxation X. Note that $\dim(\text{Conv}(X)) \leq \dim(X)$. (Why?)

Faces and Facets

Proofs

Show that $x_i \le y$ is a valid inequality and a facet of the polytope

$$X = \left\{ (\mathbf{x}, y) \in \mathbb{R}^m_+ \times \{0,1\} : \sum_{i=1}^{m} x_i \le my, x_i \le 1, i = 1, \ldots, m \right\}$$

The required result can be shown by proving the following statements.

- $\dim(\mathrm{Conv}(X)) = m + 1$
- $F_i = \left\{ (\mathbf{x}, y) \in Conv(X) : x_i = y \right\}$ is a facet, i.e., $\dim(F_i) = m$.

Faces and Facets

Proofs

Consider the following set points in Conv(X).

$$
\begin{array}{ccccc}
x_1 & x_2 & \dots & x_m & y \\
\hline
1 & 0 & \dots & 0 & 1 \\
0 & 1 & \dots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & 1 & 1 \\
0 & 0 & \dots & 0 & 1 \\
0 & 0 & \dots & 0 & 0
\end{array}
$$

Are they affinely independent? Hence, $\dim(\text{Conv}(X)) = m + 1$.

Faces and Facets

Proofs

Likewise for $F_i = \left\{(\mathbf{x}, y) \in Conv(X) : x_i = y\right\}$, the following points are affinely independent.

$$
\begin{array}{ccccc}
x_1 & x_2 & \dots & x_m & y \\
\hline
0 & 0 & \dots & 0 & 0 \\
1 & 0 & \dots & 0 & 1 \\
1 & 1 & \dots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \dots & 1 & 1
\end{array}
$$

Does this imply $\dim(F_i) = m$? We don't know if there are $m + 2$ affinely independent points in F_i. Hence, $\dim(F_i) \geq m$.

To check if $\dim(F_i) = m$, we need to show $F_i \neq$ Conv(X). (Why?) Consider $\mathbf{x} = (0, \dots, 0)$ and $y = 1$. $(\mathbf{x}, y) \in Conv(X)$ but $(\mathbf{x}, y) \notin F_i$.

The previous exercise shows how a single inequality can be shown to be facet defining.

We may also want to check if a given set of inequalities describe the convex hull of the feasible region.

There are different ways of establishing this type of results. Consider one such approach using the facility location formulation X_1.

$$X_1 = \left\{ (\mathbf{x}, y) \in \mathbb{R}_+^m \times \{0, 1\} : \sum_{i=1}^{m} x_i \leq my, x_i \leq 1, i = 1, \dots, m \right\}$$

Show that the following X_2 describes $\mathrm{Conv}(X_1)$.

$$X_2 = \left\{ (\mathbf{x}, y) \in \mathbb{R}_+^m \times \mathbb{R} : x_i \leq y, y \leq 1, i = 1, \dots, m \right\}$$

We know that $X_2 \subseteq \text{Conv}(X_1)$. It thus is enough to show that points in X_2 with fractional y are not extreme points of X_2. (Why?)

Suppose $(\mathbf{x}, y) \in X_2$ is an extreme point and fractional, i.e., $0 < y < 1$.

Consider two points $(\mathbf{0}, 0)$ and $\left(\frac{x_1}{y}, \frac{x_2}{y}, \ldots, \frac{x_m}{y}, 1\right)$. Note that both points are in X_2.

$$(\mathbf{x}, y) = (1-y)(\mathbf{0}, 0) + y\left(\frac{x_1}{y}, \frac{x_2}{y}, \ldots, \frac{x_m}{y}, 1\right)$$

However, extreme points cannot be written as the convex combination of distinct points. Hence, it cannot be a vertex of X_2.

Lifting Valid Inequalities

Lifting Valid Inequalities

Introduction

Lifting is a procedure for making valid inequalities stronger. This involves including more terms in the inequalities or adjusting the coefficients.

For example, consider the following knapsack constraint

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

Recall that $C = \{3, 4, 5, 6\}$ is a cover. The associated valid inequality is

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

What is $\dim(\text{Conv}(X))$? Is the cover inequality a facet?

Augmenting the LHS with non-negative quantities will always result in a stronger inequality. But we must ensure that it remains valid. Can we add more terms to the LHS in the above inequality?

Lifting Valid Inequalities

Cover Inequalities

Note that x_1 and x_2 have higher coefficients than that of the terms in the above valid inequality. Hence, the following inequality is also valid for X.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

Proposition

If C is a cover for $X = \left\{\mathbf{x} \in \{0,1\}^n : \sum_{j=1}^n a_j x_j \leq b\right\}$, then the extended cover inequality

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C : a_j \geq a_i \forall i \in C} x_j \leq |C| - 1$$

is also valid for X.

Is the extended cover inequality a facet? How about increasing the coefficients from 1?

Consider the following inequality generated by adding only x_1 to the original cover $C = \{3, 4, 5, 6\}$.

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

We know that $\alpha_1 = 1$ yields a valid inequality. Can we increase it?

If $x_1 = 0$, the value of α_1 does not matter. Suppose $x_1 = 1$. The constraints can be rewritten as

$$6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11 = 8$$

$$\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

What is the maximum value α_1 can take?

Lifting Valid Inequalities

Example

The maximum α_1 depends on the maximum of $x_3 + x_4 + x_5 + x_6$, which can be found using

$$\begin{aligned} \zeta = \max \ & x_3 + x_4 + x_5 + x_6 \\ \text{s.t. } & 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8 \\ & \mathbf{x} \in \{0, 1\} \end{aligned}$$

Since $\zeta = 1$, we can fix $\alpha_1 = 2$. Why did we set x_2 and x_7 to zero in the knapsack constraint?

In general, one can find values of αs for which the following is a stronger valid inequality.

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1$$

Suppose $j_1, j_2, \ldots, j_k$ is an ordering of $N \setminus C$. Consider the valid inequality

$$\alpha_{j_k} x_{j_k} + \sum_{i=1}^{k-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \leq |C| - 1$$

Suppose we have already lifted $k-1$ variables, and want to find α_{j_k}.

$$\begin{aligned} \zeta_k = \max \; & \sum_{i=1}^{k-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j \\ \text{s.t.} \; & \sum_{i=1}^{k-1} a_{j_i} x_{j_i} + \sum_{j \in C} a_j x_j \leq b - a_{j_k} \\ & \mathbf{x} \in \{0, 1\}^{|C|+k-1} \end{aligned}$$

Set $\alpha_{j_k} = |C| - 1 - \zeta_k$.

Lifting Valid Inequalities

Procedure

Using the sequential lifting procedure, find α_2 and α_7 in the earlier example.

Note that the order in which variables are lifted can result in different valid inequalities. A variable that is lifted first can have a higher coefficient.

It is possible to show that for 0–1 IPs under mild conditions, the resulting lifted inequalities are facets.

It is also possible to lift multiple coefficients simultaneously using a similar optimization problem instead of generating them sequentially.

Your Moment of Zen

Source: xkcd.com